

# Homology of Lie algebra of supersymmetries and of super Poincare Lie algebra\*

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**Abstract** We study the homology and cohomology groups of super Lie algebra of supersymmetries and of super Poincare Lie algebra. We give complete answers for (non-extended) supersymmetry in all dimensions  $\leq 11$ . For dimensions  $D = 10, 11$  we describe also the cohomology of reduction of supersymmetry Lie algebra to lower dimensions. Our methods can be applied to extended supersymmetry algebra.

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# 1 Introduction

In present paper we will analyze homology and cohomology groups of the super Lie algebra of supersymmetries and of super Poincare Lie algebra. We came to this problem studying supersymmetric deformations of maximally supersymmetric gauge theories [13]; however, this problem arises also in different situations, in particular, in supergravity [2]. In low dimensions it was studied in [5]. The cohomology of supersymmetry Lie algebra appeared also in the analysis of supersymmetric invariants in [4] (it was denoted there by the symbol  $H_t^{p,q}$ ). Some of results of present paper were derived by more elementary methods in our previous paper [14].

Let us recall the definition of Lie algebra cohomology. We start with super Lie algebra  $\mathcal{G}$  with generators  $e_A$  and structure constants  $f_{AB}^K$ . We introduce ghost variables  $C^A$  with parity opposite to the parity of generators  $e_A$  and consider the algebra  $E$  of polynomial functions of these variables. (In more invariant way we can say that  $E$  consists of polynomial functions on linear superspace  $\Pi\mathcal{G}$ .) The algebra  $E$  is graded by the degree of polynomial. We define a derivation  $d$  on  $E$  by the formula  $d = \frac{1}{2}f_{AB}^K C^A C^B \frac{\partial}{\partial C^K}$ .

This operator is a differential (i.e. it changes the parity and obeys  $d^2 = 0$ .) We define the cohomology of  $\mathcal{G}$  using this differential:

$$H^\bullet(\mathcal{G}) = \text{Ker}d / \text{Im}d.$$

The definition of homology of  $\mathcal{G}$  is dual to the definition of cohomology: instead of  $E$  we consider its dual space  $E^*$  that can be considered as the space of functions of dual ghost variables  $c_A$ ; the differential  $\partial$  on  $E^*$  is defined as an operator adjoint to  $d$ . The homology  $H_\bullet(\mathcal{G})$  is dual to the cohomology  $H^\bullet(\mathcal{G})$ . We will work with cohomology, but our results can be interpreted in the language of homology.

Notice that we can multiply cohomology classes, i.e.  $H^\bullet(\mathcal{G})$  is an algebra.

The group  $\text{Aut}(\mathcal{G})$  of automorphisms of  $\mathcal{G}$  acts on  $E$  and commutes with the differential, therefore it acts also on homology and cohomology. We will be

interested in this action. In other words we calculate cohomology as representation of this group (as  $Aut(\mathcal{G})$ -module) or as a representation of its Lie algebra  $aut(\mathcal{G})$  (as an  $aut(\mathcal{G})$ -module). For every graded module  $E$  we can define its Euler characteristic  $\chi(E)$  as a virtual module  $\sum (-1)^k E_k$  (as an alternating sum of its graded components in the sense of K-theory). Euler characteristic of graded differential module coincides with Euler characteristic of its homology. This allows us to calculate the Euler characteristic of Lie algebra cohomology as virtual representation (virtual  $Aut(\mathcal{G})$ -module) .<sup>1</sup> If the cohomology does not vanish only in one degree the Euler characteristic gives a complete answer for cohomology.

The super Lie algebra of supersymmetries has odd generators  $e_\alpha$  and even generators  $P_m$  ; the only non-trivial commutation relation is

$$[e_\alpha, e_\beta]_+ = \Gamma_{\alpha\beta}^m P_m.$$

The coefficients in this relation are expressed in terms of Dirac Gamma matrices (see e.g. [6] for mathematical introduction). The space  $E$  used in the definition of cohomology (cochain complex) consists here of polynomial functions of even ghost variables  $t^\alpha$  and odd ghost variables  $c^m$ ; the differential has the form

$$d = \frac{1}{2} \Gamma_{\alpha\beta}^m t^\alpha t^\beta \frac{\partial}{\partial c^m}. \quad (2)$$

The space  $E$  is double-graded (one can consider the degree with respect to  $t^\alpha$  and the degree with respect to  $c^m$ ). In more invariant form we can say that<sup>2</sup>

$$E = \bigoplus \text{Sym}^m S \otimes \Lambda^n V$$

where  $S$  stands for spinorial representation of orthogonal group,  $V$  denotes vector representation of this group and Gamma-matrices specify an intertwiner  $V \rightarrow \text{Sym}^2 S$ . The differential  $d$  maps  $\text{Sym}^m S \otimes \Lambda^n V$  into  $\text{Sym}^{m+2} S \otimes \Lambda^{n-1} V$ .

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<sup>1</sup>Instead of virtual modules we can talk about virtual representations of  $Aut(\mathcal{G})$  (elements of representation ring). If the group  $Aut(\mathcal{G})$  is compact the representation ring can be identified with the ring of characters).

<sup>2</sup>We use the notation  $\text{Sym}^m$  for symmetric tensor power and the notation  $\Lambda^n$  for exterior power

The description above can be applied to any dimension and to any signature of the metric used in the definition of orthogonal group, however, the choice of spinorial representation is different in different dimensions.<sup>3</sup> The group  $\mathrm{SO}(n)$  can be considered as a (subgroup) of the group of automorphisms of supersymmetry Lie algebra and therefore it acts on its cohomology. The action of  $\mathrm{SO}(n)$  is two-valued, hence it would be more precise to talk about action of its two-sheeted covering  $\mathrm{Spin}(n)$  or about action of its Lie algebra  $\mathfrak{so}(n)$ . We will work with complex representations and complex Lie algebras; this does not change the cohomology

We will consider also homology and cohomology of reduced Lie algebra of supersymmetries (or more precisely the Lie algebra of supersymmetries in dimension  $n$  reduced to the dimension  $d$ ). This algebra has the same odd generators  $e_\alpha$  as the Lie algebra of supersymmetries in dimension  $n$ , but only  $d$  even generators  $P_1, \dots, P_d$ ; the commutation relations are the same as in unreduced algebra. In this case the cohomology is a representation of  $\mathrm{Spin}(d) \times \mathrm{Spin}(n-d)$ .

The double grading on  $E$  induces double grading on cohomology. However, instead of the degrees  $m$  and  $n$  it is more convenient to use the degrees  $k = m + 2n$  and  $n$  because the differential preserves  $k$  and therefore the problem of calculation of cohomology can be solved for every  $k$  separately. It is important to notice that the differential commutes with multiplication by a polynomial depending on  $t^\alpha$ , therefore the cohomology is a module over the polynomial ring  $\mathbb{C}[t^1, \dots, t^\alpha, \dots]$ . (Moreover, it is an algebra over this ring.) The cohomology is infinite-dimensional as a vector space, but it has a finite number of generators as a  $\mathbb{C}[t^1, \dots, t^\alpha, \dots]$ -module (this follows from the fact that the polynomial ring is noetherian). One of the most important problems is the description of these generators. If cohomology classes of cocycles  $z_1, \dots, z_N$  generate the cohomology

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<sup>3</sup>Recall that orthogonal group  $\mathrm{SO}(2n)$  has two irreducible two-valued complex representations called semi-spin representations (left spinors and right spinors), the orthogonal group  $\mathrm{SO}(2n+1)$  has one irreducible two-valued complex spin representation. One says that a real representation is spinorial if after extension of scalars to  $\mathbb{C}$  it becomes a sum of spin or semi-spin representations. (We follow the terminology of [6].)

then every cohomology class can be represented by a cocycle of the form  $p_1 z_1 + \dots + p_N z_N$  where  $p_1, \dots, p_N$  belong to the polynomial ring  $\mathbb{C}[t^1, \dots, t^\alpha, \dots]$ .

Notice that the cohomology of Lie algebra of supersymmetries can be interpreted as homology of Koszul complex <sup>4</sup> corresponding to functions  $f^m(t) = \frac{1}{2} \Gamma_{\alpha\beta}^m t^\alpha t^\beta$ . This allows us to use the programs of [9] to calculate the dimensions of cohomology groups. However, we are interested in more complicated problem- in the description of decomposition of cohomology groups in direct sum of irreducible representations of the group of automorphisms *Aut* or its Lie algebra *aut*.

The paper is organized as follows. We start with the description of cohomology of Lie algebra of supersymmetries in dimension 10 (Sec.2) and in dimension 11 (Sec.3). In the next sections we describe cohomology of dimensional reductions of ten-dimensional algebra of supersymmetries (Sec.4) and of eleven-dimensional supersymmetries (Sec.5). Section 6 contains the results about Lie algebras of supersymmetries in dimensions  $\leq 9$ . Section 7 is devoted to the explanation of methods we are using. Section 8 is devoted to cohomology of super Poincare Lie algebra. In Appendix A we describe the decomposition of free resolution in direct sum of representations of the automorphism group. Appendix B gives more detail about our calculations.

## 2 D=10

We will start with ten-dimensional case; in this case the spinorial representation should be considered as one of two irreducible 16-dimensional representations of Spin(10) (the spinors are Majorana-Weyl spinors).

We will describe now the cohomology of the Lie algebra of supersymmetries in ten-dimensional case as representations of the Lie algebra  $\mathfrak{so}(10)$ . As usual the representations are labeled by coordinates of their highest weight (see e.g. [16] for details). The vector representation  $V$  has the highest weight  $[1, 0, 0, 0]$ , the

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<sup>4</sup>To define the homology of Koszul complex corresponding to functions  $f^1(t), \dots, f^n(t)$  one considers the differential  $d = f^m(t) \frac{\partial}{\partial \xi_m}$  where  $\xi_1, \dots, \xi_n$  are odd variables.

irreducible spinor representations have highest weights  $[0, 0, 0, 0, 1], [0, 0, 0, 1, 0]$ ; we assume that the highest weight of  $S$  is  $[0, 0, 0, 0, 1]$ . The description of graded component of cohomology group with gradings  $k = m + 2n$  and  $n$  is given by the formulas for  $H^{k,n}$  (for  $n \geq 6$ ,  $H^{k,n}$  vanishes)

$$H^{k,0} = [0, 0, 0, 0, k] \quad (3)$$

$$H^{k,1} = [0, 0, 0, 1, k-3] \quad (4)$$

$$H^{k,2} = [0, 0, 1, 0, k-6] \quad (5)$$

$$H^{k,3} = [0, 1, 0, 0, k-8] \quad (6)$$

$$H^{k,4} = [1, 0, 0, 0, k-10] \quad (7)$$

$$H^{k,5} = [0, 0, 0, 0, k-12] \quad (8)$$

The only special case is when  $k = 4$ , there is one additional term, a scalar, for  $H^{4,1}$ .

$$H^{4,1} = [0, 0, 0, 0, 0] \oplus [0, 0, 0, 1, 1] \quad (9)$$

The  $SO(10)$ -invariant part is in  $H^{0,0}$ ,  $H^{12,5}$ , and  $H^{4,1}$ .

The dimensions of these cohomology groups are encoded in series  $P_n(\tau) = \sum_k \dim H^{k,n} \tau^k$  (Poincare series) that can be calculated by means of [9]:

$$P_0(\tau) = \frac{\tau^3 + 5\tau^2 + 5\tau + 1}{(1-\tau)^{11}}, \quad (10)$$

$$P_1(\tau) = (16\tau^3 + 35\tau^4 - \tau^5 + 55\tau^6 - 165\tau^7 + 330\tau^8 - 462\tau^9 + 462\tau^{10} - 330\tau^{11} + 165\tau^{12} - 55\tau^{13} + 11\tau^{14} - \tau^{15})/(1-\tau)^{11}, \quad (11)$$

$$P_2(\tau) = (120\tau^6 - 120\tau^7 + 330\tau^8 - 462\tau^9 + 462\tau^{10} - 330\tau^{11} + 165\tau^{12} - 55\tau^{13} + 11\tau^{14} - \tau^{15})/(1-\tau)^{11}, \quad (12)$$

$$P_3(\tau) = \frac{45\tau^8 + 65\tau^9 + 11\tau^{10} - \tau^{11}}{(1-\tau)^{11}}, \quad (13)$$

$$P_4(\tau) = \frac{10\tau^{10} + 34\tau^{11} + 16\tau^{12}}{(1-\tau)^{11}}, \quad (14)$$

$$P_5(\tau) = \frac{\tau^{12} + 5\tau^{13} + 5\tau^{14} + \tau^{15}}{(1-\tau)^{11}} \quad (15)$$

The cohomology regarded as  $\mathbb{C}[t^1, \dots, t^\alpha, \dots]$ -module is generated by the scalar

considered as an element of  $H^{0,0}$  and by

$$\begin{aligned} [t^\alpha c_m \Gamma_{\alpha\beta}^m] &\in H^{3,1}, \\ [t^\alpha t^\beta c_m c_n \Gamma_{\alpha\beta}^{mnklr}] &\in H^{6,2}, \\ [t^\alpha t^\beta c_m c_n c_k \Gamma_{\alpha\beta}^{mnklr}] &\in H^{8,3}, \\ [t^\alpha t^\beta c_m c_n c_k c_l \Gamma_{\alpha\beta}^{mnklr}] &\in H^{10,4}, \\ [t^\alpha t^\beta c_m c_n c_k c_l c_r \Gamma_{\alpha\beta}^{mnklr}] &\in H^{12,5}. \end{aligned}$$

Here  $[a]$  denotes the cohomological class of cocycle  $a$ .

The GAMMA package [15] was used to verify the the expression above are cocycles .

### 3 D=11

Now we consider the eleven-dimensional case; in this case the spinorial representation should be considered as one irreducible 32-dimensional representations of Spin(11) (Dirac spinors). As usual we work with complex representations and complex Lie algebras. .

We will describe now the cohomology of the Lie algebra of supersymmetries in eleven-dimensional case as representations of the Lie algebra  $\mathfrak{so}(11)$ . As usual the representations are labeled by their highest weight. The vector representation  $V$  has the highest weight  $[1, 0, 0, 0, 0]$ , the irreducible spinor representations have highest weights  $[0, 0, 0, 0, 1]$ . The description of graded component of cohomology group with gradings  $k = m + 2n$  and  $n$  is given by the formulas for  $H^{k,n}$  (for  $n \geq 3$ ,  $H^{k,n}$  vanishes)

$$H^{k,0} = \bigoplus_{i=0}^{[k/2]} [0, i, 0, 0, k - 2i] \quad (16)$$

$$H^{k,1} = \bigoplus_{i=0}^{[(k-4)/2]} [1, i, 0, 0, k - 4 - 2i] \quad (17)$$

$$H^{k,2} = \bigoplus_{i=0}^{[(k-6)/2]} [0, i, 0, 0, k - 6 - 2i] \quad (18)$$

The SO(11)-invariant part is in  $H^{0,0}$  and  $H^{6,2}$ .

The dimensions of these cohomology groups are encoded in Poincare series:

$$P_0(\tau) = A(\tau) \quad (19)$$

$$P_1(\tau) = \frac{\tau^4(11 + 67\tau + 142\tau^2 + 142\tau^3 + 67\tau^4 + 11\tau^5)}{(1 - \tau)^{23}}, \quad (20)$$

$$P_2(\tau) = A(\tau)\tau^6 \quad (21)$$

where

$$A(\tau) = \frac{1 + 9\tau + 34\tau^2 + 66\tau^3 + 66\tau^4 + 34\tau^5 + 9\tau^6 + \tau^7}{(1 - \tau)^{23}} \quad (22)$$

The cohomology regarded as  $\mathbb{C}[t^1, \dots, t^\alpha, \dots]$ -module is generated by the scalar considered as an element of  $H^{0,0}$  and

$$\begin{aligned} [t^\alpha t^\beta c_m \Gamma_{\alpha\beta}^{mn}] &\in H^{4,1}, \\ [t^\alpha t^\beta c_m c_n \Gamma_{\alpha\beta}^{mn}] &\in H^{6,2}, \end{aligned}$$

## 4 Dimensional reduction from $D = 10$

Let us consider dimensional reductions of ten-dimensional Lie algebra of supersymmetries. The reduction of  $\mathfrak{su}\mathfrak{sh}_{10}$  to  $r$  dimensions has 16 odd generators (supersymmetries) and  $r$  even generators (here  $0 \leq r \leq 10$ ). Corresponding differential has the form 2 where  $\Gamma_{\alpha\beta}^m$  are ten-dimensional Dirac matrices, Greek indices take 16 values as in unreduced case, but Roman indices take only  $d$  values. The differential commutes with (two-valued) action of the group  $\mathrm{SO}(r) \times \mathrm{SO}(10 - r)$ , therefore this group acts on cohomology. The cohomology can be regarded as a module over  $\mathbb{C}[t^1, \dots, t^{16}]$ . Again cohomology is double graded; we use notation  $H^{k,n}$  for the component having degree  $m = k - 2n$  with respect to  $t$  and the degree  $n$  with respect to  $c$ . The symbol  $P_n(\tau)$  stands for the generating function  $P_n(\tau) = \sum_k \dim H^{k,n} \tau^k$  (for Poincare series). We calculate the cohomology as a representation of Lie algebra  $\mathfrak{so}(r) \times \mathfrak{so}(10 - r)$  and describe elements that generate it as a  $\mathbb{C}[t^1, \dots, t^{16}]$ -module. (We characterize the representation writing Dynkin labels of the first factor, then Dynkin labels of second factor.)



- $r = 9$ ,

$$H^{k,0} = [0, 0, 0, k], k \neq 2 \quad (23)$$

$$H^{k,1} = [0, 0, 1, k - 4] \quad (24)$$

$$H^{k,2} = [0, 1, 0, k - 6] \quad (25)$$

$$H^{k,3} = [1, 0, 0, k - 8] \quad (26)$$

$$H^{k,4} = [0, 0, 0, k - 10] \quad (27)$$

when  $k = 2$ ,

$$H^{2,0} = [0, 0, 0, 0] \oplus [0, 0, 0, 2] \quad (28)$$

Groups  $H^{k,n}$  with  $n \geq 5$  vanish. The  $\text{SO}(9)$ -invariant part is in  $H^{0,0}$ ,  $H^{10,4}$ , and  $H^{2,0}$ .

Generators

$$[t^\alpha t^\beta c_m \Gamma_{\alpha\beta}^{mnkl}] \in H^{4,1},$$

$$[t^\alpha t^\beta c_m c_n \Gamma_{\alpha\beta}^{mnkl}] \in H^{6,2},$$

$$[t^\alpha t^\beta c_m c_n c_k \Gamma_{\alpha\beta}^{mnkl}] \in H^{8,3},$$

$$[t^\alpha t^\beta c_m c_n c_k c_l \Gamma_{\alpha\beta}^{mnkl}] \in H^{10,4}.$$

Poincare series

$$P_0(\tau) = (\tau^{13} - 11\tau^{12} + 55\tau^{11} - 165\tau^{10} + 330\tau^9 - 462\tau^8 + 462\tau^7 - 330\tau^6 + 165\tau^5 - 55\tau^4 + 10\tau^3 - 6\tau^2 - 5\tau - 1)/(-1 + \tau)^{11} \quad (29)$$

$$P_1(\tau) = (84\tau^4 - 156\tau^5 + 330\tau^6 - 462\tau^7 + 462\tau^8 - 330\tau^9 + 165\tau^{10} - 55\tau^{11} + 11\tau^{12} - \tau^{13})/(1 - \tau)^{11}, \quad (30)$$

$$P_2(\tau) = \frac{36\tau^6 + 36\tau^7}{(1 - \tau)^{11}}, \quad (31)$$

$$P_3(\tau) = \frac{9\tau^8 + 29\tau^9 + 11\tau^{10} - \tau^{11}}{(1 - \tau)^{11}}, \quad (32)$$

$$P_4(\tau) = \frac{\tau^{10} + 5\tau^{11} + 5\tau^{12} + \tau^{13}}{(1 - \tau)^{11}} \quad (33)$$

- $r = 8, k > 0$ ,

$$H^{k,0} = \bigoplus_{i=1}^{k-1} [0, 0, k-i, i, k-2i] \bigoplus_{i=0}^{[k/2]} [0, 0, k-2i, 0, k] \bigoplus_{i=0}^{[k/2]} [0, 0, 0, k-2i, -k], \quad (34)$$

$$H^{k,1} = \bigoplus_{i=0}^{k-4} [0, 1, k-4-i, i, k-4-2i], \quad (35)$$

$$H^{k,2} = \bigoplus_{i=0}^{k-6} [1, 0, k-6-i, i, k-6-2i], \quad (36)$$

$$H^{k,3} = \bigoplus_{i=0}^{k-8} [0, 0, k-8-i, i, k-8-2i] \quad (37)$$

Groups  $H^{k,n}$  with  $n \geq 4$  vanish. The  $\text{SO}(8) \times \text{SO}(2)$ -invariant part is in  $H^{0,0}$ , and  $H^{8,3}$ .

Generators:

$$\begin{aligned} [t^\alpha t^\beta c_m \Gamma_{\alpha\beta}^{mnk}] &\in H^{4,1}, \\ [t^\alpha t^\beta c_m c_n \Gamma_{\alpha\beta}^{mnk}] &\in H^{6,2}, \\ [t^\alpha t^\beta c_m c_n c_k \Gamma_{\alpha\beta}^{mnk}] &\in H^{8,3}, \end{aligned}$$

Poincare series

$$P_0(\tau) = \frac{2\tau^5 - 6\tau^4 + 5\tau^3 - 7\tau^2 - 5\tau - 1}{(-1 + \tau)^{11}}, \quad (38)$$

$$P_1(\tau) = \frac{28\tau^4 + 12\tau^5 - 6\tau^6 + 2\tau^7}{(1 - \tau)^{11}}, \quad (39)$$

$$P_2(\tau) = \frac{8\tau^6 + 24\tau^7 + 6\tau^8 - 2\tau^9}{(1 - \tau)^{11}}, \quad (40)$$

$$P_3(\tau) = \frac{\tau^8 + 5\tau^9 + 5\tau^{10} + \tau^{11}}{(1 - \tau)^{11}} \quad (41)$$

- $r = 7$ ,

$$H^{k,0} = \bigoplus_{i=1}^{[k/2]} [0, i, k-2i, k-2i] \bigoplus_{i=0}^{[k/2]} [0, 0, k-2i, k] \quad (42)$$

$$H^{k,1} = \bigoplus_{i=0}^{[(k-4)/2]} [1, i, k-4-2i, k-4-2i] \quad (43)$$

$$H^{k,2} = \bigoplus_{i=0}^{[(k-6)/2]} [0, i, k-6-2i, k-6-2i] \quad (44)$$

Groups  $H^{k,n}$  with  $n \geq 3$  vanish. The  $\text{SO}(7) \times \text{SO}(3)$ -invariant part is in  $H^{0,0}$ , and  $H^{6,2}$ .

Generators:

$$\begin{aligned} [t^\alpha t^\beta c_m \Gamma_{\alpha\beta}^{mn}] &\in H^{4,1}, \\ [t^\alpha t^\beta c_m c_n \Gamma_{\alpha\beta}^{mn}] &\in H^{6,2}, \end{aligned}$$

Poincare series

$$P_0(\tau) = \frac{5\tau^5 - 7\tau^4 + 8\tau^2 + 5\tau + 1}{(-1 + \tau)^{11}}, \quad (45)$$

$$P_1(\tau) = \frac{7\tau^4 + 19\tau^5 + \tau^6 - 3\tau^7}{(1 - \tau)^{11}}, \quad (46)$$

$$P_2(\tau) = \frac{\tau^6 + 5\tau^7 + 5\tau^8 + \tau^9}{(1 - \tau)^{11}} \quad (47)$$

- $r = 6$ ,

$$\begin{aligned} H^{k,0} &= \bigoplus_{i=0}^{[k/2]} \bigoplus_{j=0}^{k-2i} [j, i, k-j-2i, j, k-j-2i] \\ &\quad \bigoplus_{i=1}^{k-1} \bigoplus_{j=\max\{0, 2i-k\}}^{i-1} [j, 0, k-2i+j, i, k-i] \end{aligned} \quad (48)$$

$$H^{k,1} = \bigoplus_{i=0}^{[(k-4)/2]} \bigoplus_{j=0}^{k-4-2i} [j, i, k-4-j-2i, j, k-4-j-2i] \quad (49)$$

Groups  $H^{k,n}$  with  $n \geq 2$  vanish. The  $\text{SO}(6) \times \text{SO}(4)$ -invariant part is in  $H^{0,0}$ , and  $H^{4,1}$ .

Generators:

$$[t^\alpha t^\beta c_m \Gamma_{\alpha\beta}^m] \in H^{4,1} \quad (50)$$

Poincare series

$$P_0(\tau) = \frac{4\tau^5 + 4\tau^4 - 5\tau^3 - 9\tau^2 - 5\tau - 1}{(-1 + \tau)^{11}}, \quad (51)$$

$$P_1(\tau) = \frac{\tau^4 + 5\tau^5 + 5\tau^6 + \tau^7}{(1 - \tau)^{11}} \quad (52)$$

- $r = 5$ ,

$$H^{k,0} = \bigoplus_{i=1}^{[k/2]} \bigoplus_{j=0}^{i-1} [j, k-2i, i, k-2i] \bigoplus_{i=0}^{[k/2]} \bigoplus_{j=0}^{[(k-2i)/2]} [i, k-2i-2j, i, k-2i-2j] \quad (53)$$

Groups  $H^{k,n}$  with  $n \geq 1$  vanish. The  $\text{SO}(5) \times \text{SO}(5)$ -invariant part lies in  $H^{k,0}$  where  $k$  is even.

Poincare series

$$P_0(\tau) = \frac{(1+\tau)^5}{(1-\tau)^{11}} \quad (54)$$

- $r = 4$ ,

$$H^{k,0} = \bigoplus_{i=0}^{[k/2]} \bigoplus_{j=0}^{k-2i} (i+1) \times [j, k-2i-j, j, i, k-2i-j] \quad (55)$$

where the coefficient  $(i+1)$  is the multiplicity. Groups  $H^{k,n}$  with  $n \geq 1$  vanish. The  $\text{SO}(4) \times \text{SO}(6)$ -invariant part is in  $H^{0,0}$ .

Poincare series

$$P_0(\tau) = \frac{(1+\tau)^4}{(1-\tau)^{12}} \quad (56)$$

- $r = 3$ ,

$$H^{k,0} = \bigoplus_{i=0}^{[k/2]} \bigoplus_{j=0}^i [k-2i, j, i-j, k-2i] \quad (57)$$

Groups  $H^{k,n}$  with  $n \geq 1$  vanish. The  $\text{SO}(3) \times \text{SO}(7)$ -invariant part is in  $H^{0,0}$ .

Poincare series

$$P_0(\tau) = \frac{(1+\tau)^3}{(1-\tau)^{13}} \quad (58)$$

- $r = 2$ ,

$$H^{k,0} = \bigoplus_{i=0}^{[k/2]} \bigoplus_{j=0}^{k-2i} [i, 0, k-2i-j, j, k-2i-2j] \quad (59)$$

Groups  $H^{k,n}$  with  $n \geq 1$  vanish. The  $\text{SO}(2) \times \text{SO}(8)$ -invariant part is in  $H^{0,0}$ .

Poincare series

$$P_0(\tau) = \frac{(1+\tau)^2}{(1-\tau)^{14}} \quad (60)$$

- $r = 1$ ,

$$H^{k,0} = \bigoplus_{i=0}^{[k/2]} [i, 0, 0, k-2i] \quad (61)$$

Groups  $H^{k,n}$  with  $n \geq 1$  vanish. The  $\text{SO}(1) \times \text{SO}(9)$ -invariant part is in  $H^{0,0}$ .

Poincare series

$$P_0(\tau) = \frac{1 + \tau}{(1 - \tau)^{15}} \quad (62)$$

For  $r \leq 5$ , the cohomology is generated by the scalar 1.

## 5 Dimensional reduction from $D = 11$

Let us consider dimensional reductions of eleven-dimensional Lie algebra of supersymmetries. The reduction of  $\mathfrak{su}\mathfrak{h}_{11}$  to  $r$  dimensions has 32 odd generators (supersymmetries) and  $r$  even generators (here  $0 \leq r \leq 11$ ). Corresponding differential has the form (2) where  $\Gamma_{\alpha\beta}^m$  are eleven-dimensional Dirac matrices, Greek indices take 32 values as in unreduced case, but Roman indices take only  $d$  values. The differential commutes with action of the group  $\mathrm{SO}(r) \times \mathrm{SO}(11-r)$ , therefore this group acts on cohomology. The cohomology can be regarded as a module over  $\mathbb{C}[t^1, \dots, t^{32}]$ . Again cohomology is double graded; we use notation  $H^{k,n}$  for the component having degree  $m = k - 2n$  with respect to  $t$  and the degree  $n$  with respect to  $c$ . The symbol  $P_n(\tau)$  stands for the generating function  $P_n(\tau) = \sum_k \dim H^{k,n} \tau^k$  (for Poincare series). We calculate the cohomology as a representation of Lie algebra  $\mathfrak{so}(r) \times \mathfrak{so}(11-r)$  and describe elements that generate it as a  $\mathbb{C}[t^1, \dots, t^{32}]$ -module.

Let us start with calculation of Euler characteristic  $\chi(H^k)$  of cohomology  $H^k = \sum_n H^{k,n}$ . By general theorem this is a virtual  $\mathfrak{so}(r) \times \mathfrak{so}(11-r)$ -module

$$\sum_n (-1)^n \mathrm{Sym}^{k-2n} S \otimes \Lambda^n V \quad (63)$$

where  $S$  and  $V$  are considered as  $\mathfrak{so}(r) \times \mathfrak{so}(11-r)$ -modules.

Cohomology for  $r = 10$  are given by the formula

$$\begin{aligned}
H^{k,0} = & \bigoplus_{i=0}^{[k/2]} \bigoplus_{j=0}^{k-2i} \bigoplus_{l=0, l \neq 2}^{[ \frac{k-2i-j}{2} ]} [0, i, 0, j, k-2i-j-2l] \\
& \bigoplus_{i=1}^{[k/2]} \bigoplus_{j=0}^{[ \frac{k-2i}{2} ]} \bigoplus_{l=0}^{k-2i-2j} [i, j, 0, l, k-2i-2j-l] \\
& \bigoplus_{i=0}^{[ \frac{k-4}{2} ]} \bigoplus_{j=0}^{k-4-2i} [0, i, 0, j, k-4-2i-j], \tag{64}
\end{aligned}$$

$$H^{k,1} = \bigoplus_{i=0}^{[ \frac{k-4}{2} ]} \bigoplus_{j=0}^i \bigoplus_{l=0}^{k-4-2i} [j, i-j, 0, l, k-4-2i-l] \tag{65}$$

$$\tag{66}$$

Groups  $H^{k,n}$  with  $n \geq 2$  vanish. The  $\text{SO}(10)$ -invariant part is in  $H^{0,0}$  and  $H^{4,1}$ .

Generators

$$[t^\alpha t^\beta c_m \Gamma_{\alpha\beta}^m] \in H^{4,1} \tag{67}$$

Poincare series

$$P_0(\tau) = \frac{(1+\tau)^{10}}{(1-\tau)^{22}} + \tau^4 A(\tau), \tag{68}$$

$$P_1(\tau) = \tau^4 A(\tau) \tag{69}$$

where  $A(\tau)$  is the Poincare series given by Eq. 22.

For  $r \leq 9$  the groups  $H^{k,n}$  with  $n \geq 1$  vanish hence Euler characteristic (63) gives a complete description of cohomology.

Poincare series

$$P_0(\tau) = \frac{(1+\tau)^r}{(1-\tau)^{32-r}} \tag{70}$$

To find the  $\mathfrak{so}(r) \times \mathfrak{so}(11-r)$ -invariant part of  $H^{k,0}$  it is sufficient to solve this problem for Euler characteristic. The conjectural answers (obtained by means of computations for  $k < 19$ ) are listed below.

$r = 1,$

$$[0, 0, 0, 0, 0] \in H^{4k,0} \tag{71}$$

$r = 2,$

$$([k/2] + 1) \times [0, 0, 0, 0, 0] \in H^{2k,0} \tag{72}$$

$r = 3,$

$$(k+1) \times [0, 0, 0, 0, 0] \in H^{4k,0} \quad (73)$$

$r = 4,$

$$\frac{(k+1)(k+2)}{2} \times [0, 0, 0, 0, 0] \in H^{4k,0} \quad (74)$$

$r = 5,$

$$\frac{([k/2]+1)([k/2]+2)([k/2]+3)}{6} \times [0, 0, 0, 0, 0] \in H^{2k,0} \quad (75)$$

$r = 6,$

$$\frac{([k/2]+1)([k/2]+2)}{2} \times [0, 0, 0, 0, 0] \in H^{2k,0} \quad (76)$$

$r = 7,$

$$(k+1) \times [0, 0, 0, 0, 0] \in H^{2k,0} \quad (77)$$

$r = 8,$

$$(2k+1) \times [0, 0, 0, 0, 0] \in H^{2k,0} \quad (78)$$

$r = 9,$

$$2 \times [0, 0, 0, 0, 0] \in H^{4k,0}, \quad (79)$$

$$[0, 0, 0, 0, 0] \in H^{4k+2,0} \quad (80)$$

where  $i \times [a, b, c, d, e]$  denotes the representation  $[a, b, c, d, e]$  with multiplicity  $i$  and  $[a]$  stands for the integer part of  $a$ .

## 6 Other dimensions

In this section we consider in detail cohomology of Lie algebra of supersymmetries in dimensions  $< 10$ . Let us begin with some general discussion of supersymmetries in various dimensions (see [6] and [12] for more detail).

We will work with complex Lie algebras. Let us start with the description of the symmetric intertwiners  $\Gamma : S^* \otimes S^* \rightarrow V$  used in the construction of supersymmetry Lie algebra in various dimensions (notice that in the construction of differential we use dual intertwiners). Recall that in even dimensions we have two irreducible spinorial representations  $\mathfrak{s}_l$  and  $\mathfrak{s}_r$ , in odd dimensions we have one irreducible spinorial representation  $\mathfrak{s}$ .

- $\dim V = 8n$

In this case we have intertwiners  $\gamma_l : \mathfrak{s}_l \otimes \mathfrak{s}_r \rightarrow V$  and  $\gamma_r : \mathfrak{s}_r \otimes \mathfrak{s}_l \rightarrow V$ .

$$S = S^* = \mathfrak{s}_l + \mathfrak{s}_r, \quad \Gamma = \gamma_l + \gamma_r, \quad \dim S = 16^n.$$

Automorphism Lie algebra  $aut = \mathfrak{so}(8n) \times \mathfrak{so}(2)$ .

- $\dim V = 8n + 1$

In this case we have symmetric intertwiner  $\gamma : \mathfrak{s} \otimes \mathfrak{s} \rightarrow V$ .

$$S = S^* = \mathfrak{s}, \quad \Gamma = \gamma, \quad \dim S = 16^n.$$

Automorphism Lie algebra  $aut = \mathfrak{so}(8n + 1)$ .

- $\dim V = 8n + 2$

In this case we have symmetric intertwiners  $\gamma_l : \mathfrak{s}_l \otimes \mathfrak{s}_l \rightarrow V$  and  $\gamma_r : \mathfrak{s}_r \otimes \mathfrak{s}_r \rightarrow V$ .

There are two possible choices of  $S$  :

$$S = \mathfrak{s}_r, S^* = \mathfrak{s}_l, \Gamma = \gamma_l; \quad S = \mathfrak{s}_l, S^* = \mathfrak{s}_r, \Gamma = \gamma_r, \dim S = 16^n.$$

Automorphism Lie algebra  $aut = \mathfrak{so}(8n + 2)$ .

- $\dim V = 8n + 3$

In this case we have symmetric intertwiner  $\gamma : \mathfrak{s} \otimes \mathfrak{s} \rightarrow V$ .

$$S = S^* = \mathfrak{s}, \quad \Gamma = \gamma, \quad \dim S = 2 \times 16^n.$$

Automorphism Lie algebra  $aut = \mathfrak{so}(8n + 3)$ .

- $\dim V = 8n + 4$

In this case we have intertwiners  $\gamma_l : \mathfrak{s}_l \otimes \mathfrak{s}_r \rightarrow V$  and  $\gamma_r : \mathfrak{s}_r \otimes \mathfrak{s}_l \rightarrow V$ .

$$S = S^* = \mathfrak{s}_l + \mathfrak{s}_r, \quad \Gamma = \gamma_l + \gamma_r, \quad \dim S = 4 \times 16^n.$$

Automorphism Lie algebra  $aut = \mathfrak{so}(8n + 4) \times \mathfrak{so}(2)$ .



- $\dim V = 8n + 5$

The intertwiner  $\gamma : \mathfrak{s} \otimes \mathfrak{s} \rightarrow V$  is antisymmetric.

$$S = S^* = \mathfrak{s} \otimes W, \Gamma = \gamma \otimes \omega, \dim S = 8 \times 16^n.$$

Here and later  $W$  stand for two-dimensional linear space with a symplectic form  $\omega$ . Automorphism Lie algebra  $aut = \mathfrak{so}(8n + 5) \times \mathfrak{sl}(2)$ .

- $\dim V = 8n + 6$

In this case we have antisymmetric intertwiners  $\gamma_l : \mathfrak{s}_l \otimes \mathfrak{s}_l \rightarrow V$  and  $\gamma_r : \mathfrak{s}_r \otimes \mathfrak{s}_r \rightarrow V$ . There are two possible choices of  $S$  :

$$S^* = \mathfrak{s}_l \otimes W, \Gamma = \gamma_l \otimes \omega; \quad S^* = \mathfrak{s}_r \otimes W, \Gamma = \gamma_r \otimes \omega, \dim S = 8 \times 16^n.$$

Automorphism Lie algebra  $aut = \mathfrak{so}(8n + 6) \times \mathfrak{sl}(2)$ .

- $\dim V = 8n + 7$

The intertwiner  $\gamma : \mathfrak{s} \otimes \mathfrak{s} \rightarrow V$  is antisymmetric.

$$S = S^* = \mathfrak{s} \otimes W, \Gamma = \gamma \otimes \omega, \dim S = 16 \times 16^n.$$

Automorphism Lie algebra  $aut = \mathfrak{so}(8n + 7) \times \mathfrak{sl}(2)$ .

One can consider also  $N$ -extended supersymmetry Lie algebra. This means that we should start with reducible spinorial representation  $S_N$  (direct sum of  $N$  copies of spinorial representation  $S$ ). Taking  $N$  copies of the intertwiner  $V \rightarrow \text{Sym}^2 S$  we obtain an intertwiner  $V \rightarrow \text{Sym}^2 S_N$ . We define the  $N$ -extended supersymmetry Lie algebra by means of this intertwiner. The Lie algebra acting on its cohomology acquires an additional factor  $\mathfrak{gl}(N)$ .

Notice that in the cases when there are two different possible choices of  $S$  (denoted by  $S_1$  and  $S_2$ ) one can talk about  $(N_1, N_2)$ -extended supersymmetry taking as a starting point a direct sum of  $N_1$  copies of  $S_1$  and  $N_2$  copies of  $S_2$ .

The description of cohomology of supersymmetry Lie algebras in dimensions 9, 8, 7 follows immediately from the description of cohomology of ten-dimensional

supersymmetry Lie algebra reduced to these dimensions. (Notice  $S$  has dimension 16 in all of these cases.)

We will describe the cohomology of the Lie algebra of supersymmetries in six-dimensional case as representations of the Lie algebra  $\mathfrak{so}(6) \times \mathfrak{sl}(2)$ . The vector representation  $V$  of  $\mathfrak{so}(6)$  has the highest weight  $[1, 0, 0]$ , the irreducible spinor representations have highest weights  $[0, 0, 1]$ ,  $[0, 1, 0]$ ; we consider for definiteness  $\mathfrak{s}_l$  with highest weight  $[0, 0, 1]$ . As a representation  $\mathfrak{so}(6) \times \mathfrak{sl}(2)$  the representation  $V$  has the weight  $[1, 0, 0, 0]$  and the representation  $S = \mathfrak{s}_l \otimes W$  has the weight  $[0, 0, 1, 1]$ . The description of graded component of cohomology group with gradings  $k = m + 2n$  and  $n$  is given by the formulas

$$H^{k,0} = [0, 0, k, k] \quad (81)$$

$$H^{k,1} = [0, 1, k-3, k-2] \quad (82)$$

$$H^{k,2} = [1, 0, k-6, k-4] \quad (83)$$

$$H^{k,3} = [0, 0, k-8, k-6] \quad (84)$$

The only special case is when  $k = 4$ , there is one additional term, a scalar, for  $H^{4,1}$ .

$$H^{4,1} = [0, 0, 0, 0] \oplus [0, 1, 1, 2] \quad (85)$$

For  $n \geq 4$ ,  $H^{k,n}$  vanishes. The cohomology considered as  $\mathbb{C}[t^1, \dots, t^\alpha, \dots]$ -module is generated by the scalar and

$$\begin{aligned} [t^\alpha c_m \Gamma_{\alpha\beta}^m] &\in H^{3,1}, \\ [t^\alpha t^\beta c_m c_n \Gamma_{\alpha\beta}^{mnk}] &\in H^{6,2}, \\ [t^\alpha t^\beta c_m c_n c_k \Gamma_{\alpha\beta}^{mnk}] &\in H^{8,3}, \\ [t^\alpha t^\beta c_m \Gamma_{\alpha\beta}^m] &\in H^{4,1}. \end{aligned}$$

Now we will describe the cohomology of the Lie algebra of supersymmetries in five-dimensional case as representations of the Lie algebra  $\mathfrak{so}(5) \times \mathfrak{sl}(2)$ . The vector representation  $V$  of  $\mathfrak{so}(5)$  has the highest weight  $[1, 0]$ , the irreducible spinorial representation has highest weight  $[0, 1]$ . As a representation  $\mathfrak{so}(5) \times \mathfrak{sl}(2)$  the representation  $V$  has the weight  $[1, 0, 0]$  and the representation  $S = \mathfrak{s} \otimes W$

has the weight  $[0, 1, 1]$ . The description of graded component of cohomology group with gradings  $k = m + 2n$  and  $n$  is given by the following formulas (for  $n \geq 3$ ,  $H^{k,n}$  vanishes)

$$H^{k,0} = [0, k, k] \quad (86)$$

$$H^{k,1} = [1, k-4, k-2] \quad (87)$$

$$H^{k,2} = [0, k-6, k-4] \quad (88)$$

The only special case is when  $k = 2$ , there is one additional term, a scalar, for  $H^{2,1}$ .

$$H^{2,1} = [0, 0, 0] \oplus [0, 2, 2] \quad (89)$$

The  $\text{SO}(5) \times \text{SL}(2)$ -invariant part is in  $H^{0,0}$ ,  $H^{6,2}$ , and  $H^{2,1}$ .

The dimensions of the cohomology groups are encoded in Poincare series:

$$P_0(\tau) = \frac{1 + 3\tau + \tau^2 - 5\tau^3 + 10\tau^4 - 10\tau^5 + 5\tau^6 - \tau^7}{(1 - \tau)^5}, \quad (90)$$

$$P_1(\tau) = \frac{15\tau^4 - 11\tau^5 + 5\tau^6 - \tau^7}{(1 - \tau)^5}, \quad (91)$$

$$P_2(\tau) = \frac{3\tau^6 + \tau^7}{(1 - \tau)^5} \quad (92)$$

The cohomology regarded as  $\mathbb{C}[t^1, \dots, t^\alpha, \dots]$ -module is generated by the scalar considered as an element of  $H^{0,0}$  and

$$[t^\alpha t^\beta c_m \Gamma_{\alpha\beta}^{mn}] \in H^{4,1},$$

$$[t^\alpha t^\beta c_m c_n \Gamma_{\alpha\beta}^{mn}] \in H^{6,2}$$

In four-dimensional case the representation  $S$  should be considered as 4-dimensional Dirac spinor.

We describe the cohomology of the Lie algebra of supersymmetries in four-dimensional case as representations of the Lie algebra  $\mathfrak{so}(4)$ . As usual the representations are labeled by their highest weight. The vector representation  $V$  has the highest weight  $[1, 1]$ , the irreducible spinor representations have highest weights  $\mathfrak{s}_l = [0, 1]$ ,  $\mathfrak{s}_r = [1, 0]$ ; we assume that  $S = \mathfrak{s}_l + \mathfrak{s}_r = [0, 1] \oplus [1, 0]$ . The

description of graded component of cohomology group with gradings  $k = m + 2n$  and  $n$  is given by the following formulas (for  $n \geq 6$ ,  $H^{k,n}$  vanishes)

$$H^{k,0} = [0, k] \oplus [k, 0] \quad (93)$$

$$H^{k,1} = [1, k-3] \oplus [k-3, 1] \quad (94)$$

$$H^{k,2} = [0, k-6] \oplus [k-6, 0] \quad (95)$$

The only special case is when  $k = 4$ , there is one additional term, a scalar, for  $H^{4,1}$ .

$$H^{4,1} = [0, 0] \oplus 2 \times [1, 1] \quad (96)$$

The  $SO(4)$ -invariant part is in  $H^{0,0}$ ,  $H^{6,2}$ , and  $H^{4,1}$ .

The dimensions of these cohomology groups are encoded in Poincare series:

$$P_0(\tau) = \frac{1 + 2\tau - \tau^2}{(1 - \tau)^2}, \quad (97)$$

$$P_1(\tau) = \frac{4\tau^3 + \tau^4 - 2\tau^5 + \tau^6}{(1 - \tau)^2}, \quad (98)$$

$$P_2(\tau) = \frac{2\tau^6}{(1 - \tau)^2} \quad (99)$$

The cohomology regarded as  $\mathbb{C}[t^1, \dots, t^\alpha, \dots]$ -module is generated by the scalar considered as an element of  $H^{0,0}$  and

$$\begin{aligned} [t^\alpha c_m \Gamma_{\alpha\beta}^m] &\in H^{3,1}, \\ [t^{\dot{\alpha}} c_m \Gamma_{\dot{\alpha}\dot{\beta}}^m] &\in H^{3,1}, \\ [t^\alpha t^\beta c_m c_n \Gamma_{\alpha\beta}^{mn}] &\in H^{6,2}, \\ [t^{\dot{\alpha}} t^{\dot{\beta}} c_m c_n \Gamma_{\dot{\alpha}\dot{\beta}}^{mn}] &\in H^{6,2}, \\ [t^\alpha t^\beta c_m \Gamma_{\alpha\beta}^m] &\in H^{4,1}, \\ [t^{\dot{\alpha}} t^{\dot{\beta}} c_m \Gamma_{\dot{\alpha}\dot{\beta}}^m] &\in H^{4,1}, \end{aligned}$$

where  $t^{\dot{\alpha}}$  and  $t^{\dot{\beta}}$  are dual Weyl spinors, and  $\alpha, \beta = 1, 2$ ,  $\dot{\alpha}, \dot{\beta} = \dot{1}, \dot{2}$ .

The cohomology generators in  $D = 4$  and  $D = 5$  were found by F. Brandt [5].

## 7 Calculations

We start with calculation of Poincare series applying [9]. However, the straightforward calculation is pretty lengthy with the computers we are using. Therefore for  $D = 10$  and  $D = 11$  we consider dimensional reduction to dimension  $r$  and we are using induction with respect to  $r$ .

Recall that the differential of  $D$ -dimensional theory reduced to dimension  $r$  has the form

$$d_r = \sum_{1 \leq m \leq r} A^m \frac{\partial}{\partial c^m}. \quad (100)$$

where  $A^m = \frac{1}{2} \Gamma_{\alpha\beta}^m t^\alpha t^\beta$  and acts in the space  $E_r$  of polynomial functions of even ghosts  $t^\alpha$  and  $r$  odd ghosts  $c^1, \dots, c^r$ . We denote corresponding cohomology by  $H_r$ . Both  $E_r$  and  $H_r$  are bigraded by the degree of even ghosts  $m$  and degree of odd ghosts  $n$ , but it is simpler to work with grading with respect to  $k = m + 2n$  and  $n$ . An element of  $E_r$  can be represented in the form  $x + yc^r$  where  $x, y \in E_{r-1}$ . Notice that

$$d_r(x + yc^r) = d_{r-1}x + A^r y + d_{r-1}yc^r.$$

The multiplication by  $A_m$  commutes with the differential, hence it induces a homomorphism  $\sigma : H_{r-1} \rightarrow H_{r-1}$ . Sending  $x \in E_{r-1}$  into  $x + 0c^m \in E_r$  (embedding  $E_{r-1}$  into  $E_r$ ) we obtain a homomorphism  $H_{r-1} \rightarrow H_r$ . Sending  $x + yc^m$  into  $y$  we get a homomorphism  $H_r \rightarrow H_{r-1}$ . It is easy to see that combining these homomorphisms we obtain exact sequence

$$H_{r-1} \rightarrow H_{r-1} \rightarrow H_r \rightarrow H_{r-1} \rightarrow H_{r-1}$$

or, taking into account the gradings,

$$\rightarrow H_{r-1}^{k,n} \rightarrow H_{r-1}^{k+2,n} \rightarrow H_r^{k+2,n} \rightarrow H_{r-1}^{k,n-1} \rightarrow H_{r-1}^{k+2,n-1} \rightarrow . \quad (101)$$

(This is the exact sequence of a pair  $(E_r, E_{r-1})$ ; we use the fact that  $E_r^{k,n}/E_{r-1}^{k,n} = E_{r-1}^{k-2,n-1}$ .) It follows immediately from this exact sequence that  $H_{r-1}^{k,n} = 0$  for  $n > n_{r-1}$  implies  $H_r^{k,n} = 0$  for  $n > n_r + 1$ . (In other words if  $n_r$  is the maximal

degree of cohomology in  $r$ -dimensional reduction then  $n_{r+1} \leq n_r + 1$ .) Applying the exact sequence (101) to the case  $n = n_r$  and assuming that  $n_{r-1} < n_r$  we obtain an isomorphism between  $H_r^{k+2, n_r}$  and a subgroup of  $H_{r-1}^{k, n_{r-1}}$  (this isomorphism can be considered as an isomorphism of  $\mathfrak{so}(r-1)$ -representations). In the cases we are interested in dimensional reductions of  $D = 10$  and  $D = 11$  the dimensions of  $H_r^{k+2, n_r}$  and  $H_{r-1}^{k, n_{r-1}}$  coincide (Poincare series are related by the formula  $P_{n_r} = \tau^2 P_{n_{r-1}}$ ). It follows that  $H_r^{k+2, n_r}$  is isomorphic to  $H_{r-1}^{k, n_{r-1}}$ . If homomorphism  $\sigma$  is injective we obtain a short exact sequence

$$0 \rightarrow H_{r-1}^{k, n} \rightarrow H_{r-1}^{k+2, n} \rightarrow H_r^{k+2, n} \rightarrow 0.$$

Calculations with [9] show that  $n_1 = \dots = n_5 = 0$  for  $D = 10$  and  $n_1 = \dots = n_9 = 0$  for  $D = 11$ . (It is sufficient to check that in corresponding dimensions the homomorphism  $\sigma$  is injective.)

To analyze  $r$ -dimensional reduction for  $r > 5, D = 10$  we notice that  $d_r$  can be considered as a sum of differentials  $d'$  and  $d''$  where

$$d' = \sum_{1 \leq m \leq 5} A^m \frac{\partial}{\partial c^m},$$

$$d'' = \sum_{5 < m \leq r} A^m \frac{\partial}{\partial c^m}.$$

(For  $r > 9, D = 11$  one should replace 5 by 9.) These differentials anticommute; this allows us to use the spectral sequence of bicomplex to calculate the cohomology of  $d_r$ . The spectral sequence of bicomplex starts with cohomology  $H(d'', H(d'))$ . Taking into account that the cohomology  $H(d') = H(d_5)$  is concentrated in degree 0 (as the cohomology  $H_5$ ) we obtain that the spectral sequence terminates. This means that one can calculate the Poincare series of  $d_r$  as the Poincare series of  $H(d'', H(d'))$  using [9]. Again applying [9] we can obtain the information about generators of cohomology; this information is sufficient to express the generators in terms of Gamma-matrices.

To calculate the cohomology as a representation of the group of automorphisms we decompose each graded component  $E^{k, n} = \text{Sym}^{k-2n} S \otimes \Lambda^n V$  of  $E$  into direct sum of irreducible representations.

For example, for  $D = 10$  spacetime, we have the cochain complex

$$\begin{aligned}
0 \xleftarrow{d_0} \text{Sym}^k S \xleftarrow{d_1} \text{Sym}^{k-2} S \otimes V \xleftarrow{d_2} \text{Sym}^{k-4} S \otimes \wedge^2 V \\
\xleftarrow{d_3} \text{Sym}^{k-6} S \otimes \wedge^3 V \xleftarrow{d_4} \text{Sym}^{k-8} S \otimes \wedge^4 V \xleftarrow{d_5} \text{Sym}^{k-10} S \otimes \wedge^5 V \\
\xleftarrow{d_6} \text{Sym}^{k-12} S \otimes \wedge^6 V \xleftarrow{d_7} \text{Sym}^{k-14} S \otimes \wedge^7 V \xleftarrow{d_8} \text{Sym}^{k-16} S \otimes \wedge^8 V \\
\xleftarrow{d_9} \text{Sym}^{k-18} S \otimes \wedge^9 V \xleftarrow{d_{10}} \text{Sym}^{k-20} S \otimes \wedge^{10} V \xleftarrow{d_{11}} 0
\end{aligned} \tag{102}$$

where for  $\text{Sym}^m S \otimes \wedge^n V$ , a grading degree defined by  $k = m + 2n$  is invariant upon cohomological differential  $d$ . All components of this complex can be regarded as representations of  $\mathfrak{so}(10)$ . We have

$$\begin{aligned}
S &= [0, 0, 0, 0, 1] \text{ (choosen) or } [0, 0, 0, 1, 0], \quad V = [1, 0, 0, 0, 0] \\
\wedge^2 V &= [0, 1, 0, 0, 0], \quad \wedge^3 V = [0, 0, 1, 0, 0], \\
\wedge^4 V &= [0, 0, 0, 1, 1], \quad \wedge^5 V = [0, 0, 0, 0, 2] \oplus [0, 0, 0, 2, 0], \\
\wedge^6 V &= \wedge^4 V, \quad \wedge^7 V = \wedge^3 V, \quad \wedge^8 V = \wedge^2 V, \quad \wedge^9 V = V, \quad \wedge^{10} V = [0, 0, 0, 0, 0],
\end{aligned} \tag{103}$$

$$\text{Sym}^k S = \bigoplus_{i=0}^{[k/2]} [i, 0, 0, 0, k - 2i] \tag{104}$$

(see [14] for the decomposition of  $\text{Sym}^m S \otimes \wedge^n V$  and for complete description of action of differential on irreducible components for supersymmetry Lie algebra in 10D and 6D.)

By the Schur's lemma an intertwiner between irreducible representations (a homomorphism of simple modules) is either zero or an isomorphism. This means that an intertwiner between non-equivalent irreducible representations always vanishes. This observation permits us to calculate the contribution of every irreducible representation to the cohomology separately.

Let us fix an irreducible representation  $A$  and the number  $k$ . We will denote by  $\nu_n$  (or by  $\nu_n^k$  if it is necessary to show the dependence of  $k$ ) the multiplicity of  $A$  in  $E^{k,n} = \text{Sym}^{k-2n} S \otimes \wedge^n V$ . The multiplicity of  $A$  in the image of  $d : E^{k,n} \rightarrow E^{k,n-1}$  will be denoted by  $\kappa_n$ , then the multiplicity of  $A$  in the kernel of this map is equal to  $\nu_n - \kappa_n$  and the multiplicity of  $A$  in the cohomology  $H^{k,n}$  is equal to  $h_n = \nu_n - \kappa_n - \kappa_{n+1}$ . It follows immediately that the multiplicity of

$A$  in virtual representation  $\sum_n (-1)^n H^{k,n}$  (in the Euler characteristic) is equal to  $\sum (-1)^n \nu_n$ . It does not depend on  $\kappa_n$ , however, to calculate the cohomology completely we should know  $\kappa_n$ .

Let us consider as an example  $A = [0, 0, 0, 0, 0]$ , the scalar representation, for dimension  $D = 10$  and arbitrary  $k$ . For all  $k \neq 4, 12$ , we have  $\nu_i = 0$ . (For small  $k$  this can be obtained by means of LiE program [8].) For  $k = 4$ , we have all  $\nu_i$  vanish except  $\nu_1 = 1$ , hence all  $\kappa_i$  vanish. The multiplicity of  $[0, 0, 0, 0, 0]$  in  $H^{4,1}$  is equal to 1, and other cohomology  $H^{4,i}$  do not contain scalar representation. For  $k = 12$ , all  $\nu_i$  vanish except  $\nu_5 = 1$ , hence  $H^{12,5}$  contains  $[0, 0, 0, 0, 0]$  with multiplicity 1, and  $H^{12,i}$  do not contain  $[0, 0, 0, 0, 0]$  for  $i \neq 5$ . This agrees with Eq. 9 and Eq. 8, respectively.

In many cases a heuristic calculation of cohomology can be based on a principle that kernel should be as small as possible; in other words, the image should be as large as possible (this is an analog of the general rule of the physics of elementary particles: Everything happens unless it is forbidden). In [7] this is called the principle of maximal propagation.<sup>5</sup> Let us illustrate this principle in the case when  $k = 9$  and  $A = [0, 1, 0, 0, 1]$  in  $10D$ . In this case  $\nu_4 = 1$ ,  $\nu_3 = 3$ ,  $\nu_2 = 1$ . If we believe in the maximal propagation, then  $\kappa_3 = 1$ ,  $\kappa_4 = 1$ , thus we have  $\nu_3 - \kappa_3 - \kappa_4 = 1$ , and  $[0, 1, 0, 0, 1]$  contributes only to  $H^{9,3}$ .

Notice, that the principle of maximal propagation sometimes does not give a definite answer. For example, in the case when  $k = 8$  in the dimension reduced to 7 from  $10D$ . Considering only the multiplicities of  $A = [0, 0, 2, 2]$ , we have sequence  $0 \rightarrow 0 \rightarrow 2 \rightarrow 5 \rightarrow 3 \rightarrow 1$ . This sequence offers two distinct possibilities even under the assumption of maximal propagation: the non-trivial  $h_n$  can be  $h_0 = 1$  only or  $h_2 = 1$  only. (We can prove that the second position is the correct choice.) In cases we are interested in one can prove using the information about Poincare series and generators that the principle of maximal propagation is working<sup>6</sup>; moreover this information permits us to resolve ambiguities in the

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<sup>5</sup> Notice that the principle of maximal propagation should be applied to the decomposition of cohomology into irreducible representations of the full automorphism group.

<sup>6</sup>We can express  $\kappa_n$  in terms of  $h_n$  and  $\nu_n$ . Obvious inequality  $0 \leq \kappa_n \leq \nu_{n-1}$  means that



application of this principle. (Sometimes it is useful to apply the remark that multiplying a coboundary by a polynomial we again obtain a coboundary.)

The only exception is the case of ten-dimensional reduction of eleven-dimensional supersymmetry Lie algebra. In this case we use the isomorphism between  $H_{10}^{k,1}$  and  $H_{11}^{k-2,2}$  that was derived from Eq. 101. This is an isomorphism of  $\mathfrak{so}(10)$ -modules; it allows us to find the decomposition of  $H_{10}^{k,1}$  from decomposition of  $H_{11}^{k-2,2}$  in irreducible representations of  $\mathfrak{so}(11)$ . From the other side we can find the virtual  $\mathfrak{so}(10)$ -character of  $H_{10}^{k,0} - H_{10}^{k,1}$  (Euler characteristic); this allows us to finish the calculation.

Let  $\rightarrow M_n \rightarrow \dots \rightarrow M_0 \rightarrow M \rightarrow 0$  denote the minimal free resolution of the module  $M = \sum_k H^{k,n}$ . Then every free module  $M_i$  can be considered as a representation of the group of automorphisms  $Aut$  (as  $Aut$ -module); it can be represented as a tensor product of  $Aut$ -module  $\mu_i$  (module of generators) and  $Aut$ -module  $\sum \text{Sym}^m S$  (polynomial algebra). It is easy to find the dimensions of  $Aut$ -modules  $\mu_i$  (the number of generators of  $M_i$ ) using [9]. The module  $M$  is graded, therefore we can talk about the grading of generators (about grading of the module  $\mu_i$  with respect to the grading in  $M$ ). This grading also can be found using [9].

The information about free resolution can be used to find the structure of  $Aut$ -module on  $\mu_i$  and therefore on  $M$ . However, we went in opposite direction: we used the information about the structure of  $Aut$ -module on  $M$  to find the structure of  $Aut$ -module on  $\mu_i$  using the formula <sup>7</sup>

$$\sum_i (-1)^i \mu_i = \left( \sum_k H^{k,n} \tau^k \right) \otimes \left( \sum (-\tau)^j \Lambda^j S \right). \quad (105)$$

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for given  $\nu_k$  the point  $(h_1, \dots, h_k, \dots)$  belongs to a convex polyhedron. Principle of maximal propagation in our understanding means that this point belongs to the boundary of this polyhedron.

<sup>7</sup>This formula follows from well known  $K$ -theory relation

$$\left( \sum \tau^m \text{Sym}^m S \right) \left( \sum (-\tau)^j \Lambda^j S \right) = 1.$$

Taking into account that parity reversal transforms symmetric power into exterior power we can understand this relation in the framework of super algebra.

The analysis of the resolution of the cohomology module is relegated to the appendix A.

## 8 Homology of super Poincare Lie algebra

The super Poincare Lie algebra can be defined as super Lie algebra spanned by supersymmetry Lie algebra and Lie algebra *aut* of its group of automorphisms *Aut*.<sup>8</sup>

To calculate the homology and cohomology of super Poincare Lie algebra we will use the following statement proved by Hochschild and Serre [11] .(It follows from Hochschild-Serre spectral sequence constructed in the same paper.)

Let  $\mathcal{P}$  denote a Lie algebra represented as a vector space as a direct sum of two subspaces  $\mathcal{L}$  and  $\mathcal{G}$ . We assume that  $\mathcal{G}$  is an ideal in  $\mathcal{P}$  and that  $\mathcal{L}$  is semisimple. It follows from the assumption that  $\mathcal{G}$  is an ideal that  $\mathcal{L}$  acts on  $\mathcal{G}$  and therefore on cohomology of  $\mathcal{G}$ ; the  $\mathcal{L}$ -invariant part of cohomology  $H^\bullet(\mathcal{G})$  will be denoted by  $H^\bullet(\mathcal{G})^\mathcal{L}$ . One can prove that

$$H^n(\mathcal{P}) = \sum_{p+q=n} H^p(\mathcal{L}) \otimes H^q(\mathcal{G})^\mathcal{L}.$$

This statement remains correct if  $\mathcal{P}$  is a super Lie algebra. We will apply it to the case when  $\mathcal{P}$  is super Poincare Lie algebra,  $\mathcal{G}$  is the Lie algebra of supersymmetries and  $\mathcal{L}$  is the Lie algebra of automorphisms or its semisimple subalgebra. (We are working with complex Lie algebras, but we can work with their real forms. The results do not change.)

Notice that it is easy to calculate the cohomology of semisimple Lie algebra  $\mathcal{L}$ ; they are described by antisymmetric tensors on  $\mathcal{L}$  that are invariant with respect to adjoint representation. One can say also that they coincide with de Rham cohomology of corresponding compact Lie group. For ten-dimensional case  $L = \mathfrak{so}_{10}$  and the compact Lie group is  $SO(10, \mathbb{R})$ . Its cohomology is a Grassmann algebra with generators of dimension 3,7,11,13 and 9. In general

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<sup>8</sup>Instead of Lie algebra of automorphisms one can take its subalgebra. For example, we can take as a subalgebra the orthogonal Lie algebra

the cohomology of the group  $\text{SO}(2r, \mathbb{R})$  is a Grassmann algebra with generators  $e_i$  having dimension  $4i - 1$  for  $i < r$  and the dimension  $2r - 1$  for  $i = r$ . The cohomology of the group  $\text{SO}(2r+1, \mathbb{R})$  is a Grassmann algebra with generators  $e_i$  having dimension  $4i - 1$  for  $i \leq r$ . The cohomology of Lie algebra  $\mathfrak{sl}(n)$  coincide with the cohomology of compact Lie group  $\text{SU}(n)$ ; they form a Grassmann algebra with generators of dimension  $3, 5, \dots, 2n - 1$ .

As we have seen only  $\mathcal{L}$ -invariant part of cohomology of Lie algebra of supersymmetries contributes to the cohomology of super Poincare algebra. For  $D = 10$  this means that the only contribution comes from  $(m, n) = (0, 0)$ ,  $(m, n) = (2, 1)$  and  $(m, n) = (2, 5)$ , for  $D = 11$  the only contribution comes from  $(m, n) = (0, 0)$  and  $(m, n) = (2, 2)$ , for  $D = 6$  the only contribution comes from  $(m, n) = (0, 0)$  and  $(m, n) = (2, 1)$ . (Here  $m = k - 2n$  denotes the grading with respect to even ghosts  $t^\alpha$  and  $n$  the grading with respect to odd ghosts  $c_m$ .)

Cocycles representing cohomology classes of super Poincare algebra can be written in the form  $\rho \otimes h$ , where  $\rho$  is an invariant antisymmetric tensor with respect to adjoint representation of  $\mathfrak{aut}$  and  $h$  is 1 or

$$t^\alpha t^\beta c_m c_n \Gamma_{\alpha\beta}^{mn} \text{ for } D = 11 \quad (106)$$

$$t^\alpha t^\beta c_m \Gamma_{\alpha\beta}^m, \quad t^\alpha t^\beta c_m c_n c_k c_l c_r \Gamma_{\alpha\beta}^{mnklr} \text{ for } D = 10 \quad (107)$$

$$t^\alpha t^\alpha, \quad t^\alpha t^\beta c_m c_n c_k c_l \Gamma_{\alpha\beta}^{mnkl} \text{ for } D = 9 \quad (108)$$

$$t^\alpha t^\beta c_m c_n c_k \Gamma_{\alpha\beta}^{mnk} \text{ for } D = 8 \quad (109)$$

$$t^\alpha t^\beta c_m c_n \Gamma_{\alpha\beta}^{mn} \text{ for } D = 7 \quad (110)$$

$$t^\alpha t^\beta c_m \Gamma_{\alpha\beta}^m, \quad t^\alpha t^\beta c_m c_n c_k \Gamma_{\alpha\beta}^{mnk} \text{ for } D = 6 \quad (111)$$

$$t^\alpha t^\beta c_m c_n \Gamma_{\alpha\beta}^{mn} \text{ for } D = 5 \quad (112)$$

$$t^\alpha t^\beta c_m \Gamma_{\alpha\beta}^m, t^{\dot{\alpha}} t^{\dot{\beta}} c_m \Gamma_{\dot{\alpha}\dot{\beta}}^m, t^\alpha t^\beta c_m c_n \Gamma_{\alpha\beta}^{mn}, t^{\dot{\alpha}} t^{\dot{\beta}} c_m c_n \Gamma_{\dot{\alpha}\dot{\beta}}^{mn} \text{ for } D = 4. \quad (113)$$

Here Greek indices (i.e. spinor indices) take values  $1, 2, \dots, \dim S$  and Roman indices (i.e. vector indices) take values  $1, 2, \dots, D$ , and  $\dim S$  is defined in Section.6. The only exception is for  $D = 4$ , the Greek indices  $\alpha, \beta$  take values

$1, 2, \dots, \dim S$ , and the dotted Greek indices  $\dot{\alpha}, \dot{\beta}$  take values  $\dot{1}, \dot{2}, \dots, \dim S$ . Notice that in these formulas Gamma matrices and summation range depend on the choice of dimension.

The general definition of super Poincare algebra can be applied also to reduced supersymmetry Lie algebra. For  $D = 10$  and  $D = 11$  the role of super Poincare Lie algebra is played by the semidirect product of reduced supersymmetry Lie algebra and  $\mathfrak{so}(r) \times \mathfrak{so}(D - r)$ . The information about invariant elements provided in Sections 4 and 5 permits us to describe cohomology of this generalization of super Poincare algebra.

## A Resolution of the cohomology modules

One can find a minimal free resolution of the  $R$ -module  $\sum_k H^{k,n} = M$ . (Here  $R = \mathbb{C}[t^1, \dots, t^\alpha, \dots] = \sum_m \text{Sym}^m S$ .) The reader may wish to consult [1] on this subject. The free resolution has the form

$$\dots \rightarrow M_i \rightarrow \dots \rightarrow M_0 \rightarrow M \rightarrow 0$$

where  $M_i = \mu_i \otimes R$ , and

$\mu_0$  - generators of  $M$ ;

$\mu_1$  - relations between generators of  $M$ ;

$\mu_2$  - relations between relations ;

$\dots$

We give the structure of  $\mu_i$  as *aut*-module and its grading in the case of  $r$ -dimensional reduction of ten-dimensional Lie algebra of supersymmetries (in the case when  $D = r + (10 - r)$ ). Recall that *aut* denotes the Lie algebra of the group of automorphisms *Aut*.

- D=10+0, n=0

$$\mu_0 = [0, 0, 0, 0, 0], \dim(\mu_0) = 1, \deg(\mu_0) = 0;$$

$$\mu_1 = [1, 0, 0, 0, 0], \dim(\mu_1) = 10, \deg(\mu_1) = 2;$$

$$\mu_2 = [0, 0, 0, 1, 0], \dim(\mu_2) = 16, \deg(\mu_2) = 3;$$

$$\mu_3 = [0, 0, 0, 0, 1], \dim(\mu_3) = 16, \deg(\mu_3) = 5;$$

$$\mu_4 = [1, 0, 0, 0, 0], \dim(\mu_4) = 10, \deg(\mu_4) = 6;$$

$$\mu_5 = [0, 0, 0, 0, 0], \dim(\mu_5) = 1, \deg(\mu_5) = 8.$$

- D=10+0, n=1

$$\mu_0 = [0, 0, 0, 1, 0], \dim(\mu_0) = 16, \deg(\mu_0) = 3;$$

$$\mu_1 = \mu_1' + \mu_1'',$$

$$\mu_1' = [0, 1, 0, 0, 0], \dim(\mu_1') = 45, \deg(\mu_1') = 4;$$

$$\mu_1'' = [0, 0, 0, 0, 1], \dim(\mu_1'') = 16, \deg(\mu_1'') = 5;$$

$$\mu_2 = 2 \times [0, 0, 1, 0, 0] + [1, 0, 0, 0, 0], \dim(\mu_2) = 250, \deg(\mu_2) = 6;$$

$$\mu_3 = [0, 0, 0, 1, 0] + [0, 1, 0, 1, 0] + [1, 0, 0, 0, 1], \dim(\mu_3) = 720, \deg(\mu_3) = 7;$$

$$\mu_4 = \mu_4' + \mu_4'',$$

$$\mu_4' = [0, 2, 0, 0, 0] + [1, 0, 0, 2, 0] + [2, 0, 0, 0, 0], \dim(\mu_4') = 1874, \deg(\mu_4') = 8;$$

$$\mu_4'' = [0, 0, 0, 0, 1], \dim(\mu_4'') = 16, \deg(\mu_4'') = 9;$$

$$\mu_5 = \mu_5' + \mu_5'',$$

$$\mu_5' = [0, 0, 0, 0, 1] + [0, 0, 0, 0, 1] + [0, 0, 0, 3, 0] + [1, 1, 0, 1, 0], \dim(\mu_5') = 4352, \deg(\mu_5') = 9;$$

$$\mu_5'' = [1, 0, 0, 0, 0], \dim(\mu_5'') = 9, \deg(\mu_5'') = 10;$$

$$\mu_6 = [0, 1, 0, 2, 0] + [2, 0, 1, 0, 0], \dim(\mu_6) = 8008, \deg(\mu_6) = 10;$$

$$\mu_7 = [1, 0, 1, 1, 0] + [3, 0, 0, 0, 1], \dim(\mu_7) = 11440, \deg(\mu_7) = 11;$$

$$\mu_8 = [0, 0, 2, 0, 0] + [2, 0, 0, 1, 1] + [4, 0, 0, 0, 0], \dim(\mu_8) = 12870, \deg(\mu_8) = 12;$$

$$\mu_9 = [1, 0, 1, 0, 1] + [3, 0, 0, 1, 0], \dim(\mu_9) = 11440, \deg(\mu_9) = 13;$$

$$\mu_{10} = [0, 1, 0, 0, 2] + [2, 0, 1, 0, 0], \dim(\mu_{10}) = 8008, \deg(\mu_{10}) = 14;$$

$$\mu_{11} = [0, 0, 0, 0, 3] + [1, 1, 0, 0, 1], \dim(\mu_{11}) = 4368, \deg(\mu_{11}) = 15;$$

$$\mu_{12} = [0, 2, 0, 0, 0] + [1, 0, 0, 0, 2], \dim(\mu_{12}) = 1820, \deg(\mu_{12}) = 16;$$

$$\mu_{13} = [0, 1, 0, 0, 1], \dim(\mu_{13}) = 560, \deg(\mu_{13}) = 17;$$

$$\mu_{14} = [0, 0, 1, 0, 0], \dim(\mu_{14}) = 120, \deg(\mu_{14}) = 18;$$

$$\mu_{15} = [0, 0, 0, 1, 0], \dim(\mu_{15}) = 16, \deg(\mu_{15}) = 19;$$

$$\mu_{16} = [0, 0, 0, 0, 0], \dim(\mu_{16}) = 1, \deg(\mu_{16}) = 20.$$

- D=10+0, n=2

$$\mu_0 = [0, 0, 1, 0, 0], \dim(\mu_0) = 120, \deg(\mu_0) = 6;$$

$$\mu_1 = [0, 0, 0, 1, 0] + [0, 1, 0, 1, 0] + [1, 0, 0, 0, 1], \dim(\mu_1) = 720, \deg(\mu_1) = 7;$$

$$\mu_2 = [0, 0, 0, 0, 0] + [0, 0, 0, 1, 1] + [0, 1, 0, 0, 0] + [0, 2, 0, 0, 0] + [1, 0, 0, 2, 0] + [2, 0, 0, 0, 0], \dim(\mu_2) = 2130, \deg(\mu_2) = 8;$$

$$\mu_3 = \mu_3' + \mu_3'',$$

$$\mu_3' = [0, 0, 0, 3, 0] + [1, 0, 0, 1, 0] + [1, 1, 0, 1, 0], \dim(\mu_3') = 4512, \deg(\mu_3') = 9;$$

$$\mu_3'' = [0, 0, 0, 0, 2] + [1, 0, 0, 0, 0], \dim(\mu_3'') = 136, \deg(\mu_3'') = 10;$$

$$\mu_4 = \mu_4' + \mu_4'',$$

$$\mu_4' = [0, 1, 0, 2, 0] + [2, 0, 1, 0, 0], \dim(\mu_4') = 8008, \deg(\mu_4') = 10;$$

$$\mu_4'' = [0, 0, 0, 1, 0] + [1, 0, 0, 0, 1], \dim(\mu_4'') = 160, \deg(\mu_4'') = 11;$$

$$\mu_5 = \mu_5' + \mu_5'',$$

$$\mu_5' = [1, 0, 1, 1, 0] + [3, 0, 0, 0, 1], \dim(\mu_5') = 11440, \deg(\mu_5') = 11;$$

$$\mu_5'' = [0, 1, 0, 0, 0], \dim(\mu_5'') = 45, \deg(\mu_5'') = 12;$$

$$\mu_6 = [0, 0, 2, 0, 0] + [2, 0, 0, 1, 1] + [4, 0, 0, 0, 0], \dim(\mu_6) = 12870, \deg(\mu_6) = 12;$$

$$\mu_7 = [1, 0, 1, 0, 1] + [3, 0, 0, 1, 0], \dim(\mu_7) = 11440, \deg(\mu_7) = 13;$$

$$\mu_8 = [0, 1, 0, 0, 2] + [2, 0, 1, 0, 0], \dim(\mu_8) = 8008, \deg(\mu_8) = 14;$$

$$\mu_9 = [0, 0, 0, 0, 3] + [1, 1, 0, 0, 1], \dim(\mu_9) = 4368, \deg(\mu_9) = 15;$$

$$\mu_{10} = [0, 2, 0, 0, 0] + [1, 0, 0, 0, 2], \dim(\mu_{10}) = 1820, \deg(\mu_{10}) = 16;$$

$$\mu_{11} = [0, 1, 0, 0, 1], \dim(\mu_{11}) = 560, \deg(\mu_{11}) = 17;$$

$$\mu_{12} = [0, 0, 1, 0, 0], \dim(\mu_{12}) = 120, \deg(\mu_{12}) = 18;$$

$$\mu_{13} = [0, 0, 0, 1, 0], \dim(\mu_{13}) = 16, \deg(\mu_{13}) = 19;$$

$$\mu_{14} = [0, 0, 0, 0, 0], \dim(\mu_{14}) = 1, \deg(\mu_{14}) = 20.$$

- D=10+0, n=3

$$\mu_0 = [0, 1, 0, 0, 0], \dim(\mu_0) = 45, \deg(\mu_0) = 8;$$

$$\mu_1 = [0, 0, 0, 0, 1] + [1, 0, 0, 1, 0], \dim(\mu_1) = 160, \deg(\mu_1) = 9;$$

$$\mu_2 = \mu_2' + \mu_2'',$$

$$\mu_2' = [0, 0, 0, 2, 0] + [1, 0, 0, 0, 0], \dim(\mu_2') = 136, \deg(\mu_2') = 10;$$

$$\mu_2'' = [1, 0, 0, 0, 1], \dim(\mu_2'') = 144, \deg(\mu_2'') = 11;$$

$$\mu_3 = [0, 0, 0, 0, 0] + [0, 0, 0, 1, 1] + [0, 1, 0, 0, 0] + [2, 0, 0, 0, 0], \dim(\mu_3) = 310, \deg(\mu_3) = 12;$$

$$\mu_4 = [0, 0, 0, 0, 1] + [1, 0, 0, 1, 0], \dim(\mu_4) = 160, \deg(\mu_4) = 13;$$

$$\mu_5 = [0, 0, 0, 1, 0], \dim(\mu_5) = 16, \deg(\mu_5) = 15;$$

$$\mu_6 = [0, 0, 0, 0, 0], \dim(\mu_6) = 1, \deg(\mu_6) = 16.$$

- D=10+0, n=4

$$\mu_0 = [1, 0, 0, 0, 0], \dim(\mu_0) = 10, \deg(\mu_0) = 10;$$

$$\mu_1 = \mu_1' + \mu_1'',$$

$$\mu_1' = [0, 0, 0, 1, 0], \dim(\mu_1') = 16, \deg(\mu_1') = 11;$$

$$\mu_1'' = [2, 0, 0, 0, 0], \dim(\mu_1'') = 54, \deg(\mu_1'') = 12;$$

$$\mu_2 = [0, 0, 0, 0, 1] + [1, 0, 0, 1, 0], \dim(\mu_2) = 160, \deg(\mu_2) = 13;$$

$$\mu_3 = [0, 0, 1, 0, 0] + [1, 0, 0, 0, 0], \dim(\mu_3) = 130, \deg(\mu_3) = 14;$$

$$\mu_4 = [0, 0, 0, 0, 0] + [0, 1, 0, 0, 0], \dim(\mu_4) = 46, \deg(\mu_4) = 16;$$

$$\mu_5 = [0, 0, 0, 0, 1], \dim(\mu_5) = 16, \deg(\mu_5) = 17.$$

- D=10+0, n=5

$$\mu_0 = [0, 0, 0, 0, 0], \dim(\mu_0) = 1, \deg(\mu_0) = 12;$$

$$\mu_1 = [1, 0, 0, 0, 0], \dim(\mu_1) = 10, \deg(\mu_1) = 14;$$

$$\mu_2 = [0, 0, 0, 1, 0], \dim(\mu_2) = 16, \deg(\mu_2) = 15;$$

$$\mu_3 = [0, 0, 0, 0, 1], \dim(\mu_3) = 16, \deg(\mu_3) = 17;$$

$$\mu_4 = [1, 0, 0, 0, 0], \dim(\mu_4) = 10, \deg(\mu_4) = 18;$$

$$\mu_5 = [0, 0, 0, 0, 0], \dim(\mu_5) = 1, \deg(\mu_5) = 20.$$

- D=9+1, n=0

$$\mu_0 = [0, 0, 0, 0], \dim(\mu_0) = 1, \deg(\mu_0) = 0;$$

$$\mu_1 = [1, 0, 0, 0], \dim(\mu_1) = 9, \deg(\mu_1) = 2;$$

$$\mu_2 = [0, 0, 1, 0] + [0, 1, 0, 0], \dim(\mu_2) = 120, \deg(\mu_2) = 4;$$

$$\mu_3 = [0, 0, 0, 1] + [0, 1, 0, 1] + [1, 0, 0, 1], \dim(\mu_3) = 576, \deg(\mu_3) = 5;$$

$$\begin{aligned} \mu_4 = & [0, 0, 0, 0] + [0, 0, 0, 2] + [0, 2, 0, 0] + [1, 0, 0, 0] + [1, 0, 0, 2] + [1, 1, 0, 0] + \\ & + [2, 0, 0, 0], \dim(\mu_4) = 1830, \deg(\mu_4) = 6; \end{aligned}$$

$$\mu_5 = \mu_5' + \mu_5'',$$

$$\mu_5' = [0, 0, 0, 3] + [0, 1, 0, 1] + [1, 0, 0, 1] + [1, 1, 0, 1] + [2, 0, 0, 1], \dim(\mu_5') = 4368, \deg(\mu_5') = 7;$$

$$\mu_5'' = [0, 0, 0, 0], \dim(\mu_5'') = 1, \deg(\mu_5'') = 8;$$

$$\begin{aligned} \mu_6 = & [0, 0, 1, 0] + [0, 1, 0, 0] + [0, 1, 0, 2] + [1, 0, 0, 2] + [1, 0, 1, 0] + [1, 1, 0, 0] + \\ & + [2, 0, 1, 0] + [2, 1, 0, 0], \dim(\mu_6) = 8008, \deg(\mu_6) = 8; \end{aligned}$$

$$\begin{aligned} \mu_7 = & [0, 0, 0, 1] + [0, 0, 1, 1] + [0, 1, 0, 1] + [1, 0, 0, 1] + [1, 0, 1, 1] + [1, 1, 0, 1] + \\ & + [2, 0, 0, 1] + [3, 0, 0, 1], \dim(\mu_7) = 11440, \deg(\mu_7) = 9; \end{aligned}$$

$$\begin{aligned} \mu_8 = & [0, 0, 0, 0] + [0, 0, 0, 2] + [0, 0, 1, 0] + [0, 0, 2, 0] + [0, 1, 1, 0] + [0, 2, 0, 0] + \\ & + [1, 0, 0, 0] + [1, 0, 0, 2] + [1, 0, 1, 0] + [2, 0, 0, 0] + [2, 0, 0, 2] + [2, 0, 1, 0] + \\ & + [3, 0, 0, 0] + [4, 0, 0, 0], \dim(\mu_8) = 12870, \deg(\mu_8) = 10; \end{aligned}$$



$$\begin{aligned}
\mu_9 &= [0, 0, 0, 1] + [0, 0, 1, 1] + [0, 1, 0, 1] + [1, 0, 0, 1] + [1, 0, 1, 1] + [1, 1, 0, 1] + \\
&\quad + [2, 0, 0, 1] + [3, 0, 0, 1], \dim(\mu_9) = 11440, \deg(\mu_9) = 11; \\
\mu_{10} &= [0, 0, 1, 0] + [0, 1, 0, 0] + [0, 1, 0, 2] + [1, 0, 0, 2] + [1, 0, 1, 0] + [1, 1, 0, 0] + \\
&\quad + [2, 0, 1, 0] + [2, 1, 0, 0], \dim(\mu_{10}) = 8008, \deg(\mu_{10}) = 12; \\
\mu_{11} &= [0, 0, 0, 3] + [0, 1, 0, 1] + [1, 0, 0, 1] + [1, 1, 0, 1] + [2, 0, 0, 1], \dim(\mu_{11}) = 4368, \deg(\mu_{11}) = 13; \\
\mu_{12} &= [0, 0, 0, 2] + [0, 2, 0, 0] + [1, 0, 0, 2] + [1, 1, 0, 0] + [2, 0, 0, 0], \dim(\mu_{12}) = 1820, \deg(\mu_{12}) = 14; \\
\mu_{13} &= [0, 1, 0, 1] + [1, 0, 0, 1], \dim(\mu_{13}) = 560, \deg(\mu_{13}) = 15; \\
\mu_{14} &= [0, 0, 1, 0] + [0, 1, 0, 0], \dim(\mu_{14}) = 120, \deg(\mu_{14}) = 16; \\
\mu_{15} &= [0, 0, 0, 1], \dim(\mu_{15}) = 16, \deg(\mu_{15}) = 17; \\
\mu_{16} &= [0, 0, 0, 0], \dim(\mu_{16}) = 1, \deg(\mu_{16}) = 18.
\end{aligned}$$

• D=9+1, n=1

$$\begin{aligned}
\mu_0 &= [0, 0, 1, 0], \dim(\mu_0) = 84, \deg(\mu_0) = 4; \\
\mu_1 &= [0, 0, 0, 1] + [0, 1, 0, 1] + [1, 0, 0, 1], \dim(\mu_1) = 576, \deg(\mu_1) = 5; \\
\mu_2 &= [0, 0, 0, 0] + [0, 0, 0, 2] + [0, 0, 1, 0] + [0, 1, 0, 0] + [0, 2, 0, 0] + [1, 0, 0, 0] + \\
&\quad + [1, 0, 0, 2] + [1, 1, 0, 0] + [2, 0, 0, 0], \dim(\mu_2) = 1950, \deg(\mu_2) = 6; \\
\mu_3 &= [0, 0, 0, 1] + [0, 0, 0, 3] + [0, 1, 0, 1] + 2 \times [1, 0, 0, 1] + [1, 1, 0, 1] + [2, 0, 0, 1], \\
&\quad \dim(\mu_3) = 4512, \deg(\mu_3) = 7; \\
\mu_4 &= \mu_4' + \mu_4'', \\
\mu_4' &= [0, 0, 1, 0] + [0, 1, 0, 0] + [0, 1, 0, 2] + [1, 0, 0, 2] + [1, 0, 1, 0] + [1, 1, 0, 0] + \\
&\quad + [2, 0, 0, 0] + [2, 0, 1, 0] + [2, 1, 0, 0], \dim(\mu_4') = 8052, \deg(\mu_4') = 8; \\
\mu_4'' &= [0, 0, 0, 1], \dim(\mu_4'') = 16, \deg(\mu_4'') = 9; \\
\mu_5 &= \mu_5' + \mu_5'', \\
\mu_5' &= [0, 0, 0, 1] + [0, 0, 1, 1] + [0, 1, 0, 1] + [1, 0, 0, 1] + [1, 0, 1, 1] + [1, 1, 0, 1] + \\
&\quad + [2, 0, 0, 1] + [3, 0, 0, 1], \dim(\mu_5') = 11440, \deg(\mu_5') = 9; \\
\mu_5'' &= [1, 0, 0, 0], \dim(\mu_5'') = 9, \deg(\mu_5'') = 10;
\end{aligned}$$

$$\begin{aligned}
\mu_6 &= [0, 0, 0, 0] + [0, 0, 0, 2] + [0, 0, 1, 0] + [0, 0, 2, 0] + [0, 1, 1, 0] + [0, 2, 0, 0] + \\
&\quad + [1, 0, 0, 0] + [1, 0, 0, 2] + [1, 0, 1, 0] + [2, 0, 0, 0] + [2, 0, 0, 2] + [2, 0, 1, 0] + \\
&\quad + [3, 0, 0, 0] + [4, 0, 0, 0], \dim(\mu_6) = 12870, \deg(\mu_6) = 10; \\
\mu_7 &= [0, 0, 0, 1] + [0, 0, 1, 1] + [0, 1, 0, 1] + [1, 0, 0, 1] + [1, 0, 1, 1] + [1, 1, 0, 1] + \\
&\quad + [2, 0, 0, 1] + [3, 0, 0, 1], \dim(\mu_7) = 11440, \deg(\mu_7) = 11; \\
\mu_8 &= [0, 0, 1, 0] + [0, 1, 0, 0] + [0, 1, 0, 2] + [1, 0, 0, 2] + [1, 0, 1, 0] + [1, 1, 0, 0] + \\
&\quad + [2, 0, 1, 0] + [2, 1, 0, 0], \dim(\mu_8) = 8008, \deg(\mu_8) = 12; \\
\mu_9 &= [0, 0, 0, 3] + [0, 1, 0, 1] + [1, 0, 0, 1] + [1, 1, 0, 1] + [2, 0, 0, 1], \dim(\mu_9) = 4368, \deg(\mu_9) = 13; \\
\mu_{10} &= [0, 0, 0, 2] + [0, 2, 0, 0] + [1, 0, 0, 2] + [1, 1, 0, 0] + [2, 0, 0, 0], \dim(\mu_{10}) = 1820, \deg(\mu_{10}) = 14; \\
\mu_{11} &= [0, 1, 0, 1] + [1, 0, 0, 1], \dim(\mu_{11}) = 560, \deg(\mu_{11}) = 15; \\
\mu_{12} &= [0, 0, 1, 0] + [0, 1, 0, 0], \dim(\mu_{12}) = 120, \deg(\mu_{12}) = 16; \\
\mu_{13} &= [0, 0, 0, 1], \dim(\mu_{13}) = 16, \deg(\mu_{13}) = 17; \\
\mu_{14} &= [0, 0, 0, 0], \dim(\mu_{14}) = 1, \deg(\mu_{14}) = 18.
\end{aligned}$$

- D=9+1, n=2

$$\begin{aligned}
\mu_0 &= [0, 1, 0, 0], \dim(\mu_0) = 36, \deg(\mu_0) = 6; \\
\mu_1 &= [0, 0, 0, 1] + [1, 0, 0, 1], \dim(\mu_1) = 144, \deg(\mu_1) = 7; \\
\mu_2 &= [0, 0, 0, 0] + [0, 0, 0, 2] + [1, 0, 0, 0] + [2, 0, 0, 0], \dim(\mu_2) = 180, \deg(\mu_2) = 8; \\
\mu_3 &= [0, 0, 0, 0] + [0, 0, 0, 2] + [1, 0, 0, 0] + [2, 0, 0, 0], \dim(\mu_3) = 180, \deg(\mu_3) = 10; \\
\mu_4 &= [0, 0, 0, 1] + [1, 0, 0, 1], \dim(\mu_4) = 144, \deg(\mu_4) = 11; \\
\mu_5 &= [0, 1, 0, 0], \dim(\mu_5) = 36, \deg(\mu_5) = 12.
\end{aligned}$$

- D=9+1, n=3

$$\mu_0 = [1, 0, 0, 0], \dim(\mu_0) = 9, \deg(\mu_0) = 8;$$

$$\mu_1 = \mu_1' + \mu_1'',$$

$$\mu_1' = [0, 0, 0, 1], \dim(\mu_1') = 16, \deg(\mu_1') = 9;$$

$$\mu_1'' = [2, 0, 0, 0], \dim(\mu_1'') = 44, \deg(\mu_1'') = 10;$$

$$\mu_2 = [0, 0, 0, 1] + [1, 0, 0, 1], \dim(\mu_2) = 144, \deg(\mu_2) = 11;$$

$$\mu_3 = [0, 0, 0, 0] + [0, 0, 1, 0] + [0, 1, 0, 0] + [1, 0, 0, 0], \dim(\mu_3) = 130, \deg(\mu_3) = 12;$$

$$\mu_4 = \mu_4' + \mu_4'',$$

$$\mu_4' = [0, 0, 0, 1], \dim(\mu_4') = 16, \deg(\mu_4') = 13;$$

$$\mu_4'' = [0, 1, 0, 0], \dim(\mu_4'') = 36, \deg(\mu_4'') = 14;$$

$$\mu_5 = [0, 0, 0, 1], \dim(\mu_5) = 16, \deg(\mu_5) = 15;$$

$$\mu_6 = [0, 0, 0, 0], \dim(\mu_6) = 1, \deg(\mu_6) = 16.$$

- D=9+1, n=4

$$\mu_0 = [0, 0, 0, 0], \dim(\mu_0) = 1, \deg(\mu_0) = 10;$$

$$\mu_1 = [0, 0, 0, 0] + [1, 0, 0, 0], \dim(\mu_1) = 10, \deg(\mu_1) = 12;$$

$$\mu_2 = [0, 0, 0, 1], \dim(\mu_2) = 16, \deg(\mu_2) = 13;$$

$$\mu_3 = [0, 0, 0, 1], \dim(\mu_3) = 16, \deg(\mu_3) = 15;$$

$$\mu_4 = [0, 0, 0, 0] + [1, 0, 0, 0], \dim(\mu_4) = 10, \deg(\mu_4) = 16;$$

$$\mu_5 = [0, 0, 0, 0], \dim(\mu_5) = 1, \deg(\mu_5) = 18.$$

- D=8+2, n=0

$$\mu_0 = [0, 0, 0, 0, 0], \dim(\mu_0) = 1, \deg(\mu_0) = 0;$$

$$\mu_1 = [1, 0, 0, 0, 0], \dim(\mu_1) = 8, \deg(\mu_1) = 2;$$

$$\mu_2 = [0, 1, 0, 0, 0], \dim(\mu_2) = 56, \deg(\mu_2) = 4;$$

$$\mu_3 = [0, 0, 0, 1, -1] + [0, 0, 1, 0, 1] + [1, 0, 0, 1, 1] + [1, 0, 1, 0, -1], \dim(\mu_3) = 128, \deg(\mu_3) = 5;$$

$$\begin{aligned} \mu_4 = & [0, 0, 0, 0, -2] + [0, 0, 0, 0, 2] + [0, 0, 0, 2, 2] + [0, 0, 2, 0, -2] + [1, 0, 0, 0, 0] + \\ & + [2, 0, 0, 0, -2] + [2, 0, 0, 0, 2], \dim(\mu_4) = 150, \deg(\mu_4) = 6; \end{aligned}$$

$$\mu_5 = \mu_5' + \mu_5'',$$

$$\mu_5' = [1, 0, 0, 1, 3] + [1, 0, 1, 0, -3], \dim(\mu_5') = 112, \deg(\mu_5') = 7;$$

$$\mu_5'' = [0, 0, 0, 0, 0], \dim(\mu_5'') = 1, \deg(\mu_5'') = 8;$$

$$\mu_6 = [0, 1, 0, 0, -4] + [0, 1, 0, 0, 4], \dim(\mu_6) = 56, \deg(\mu_6) = 8;$$

$$\mu_7 = [0, 0, 0, 1, -5] + [0, 0, 1, 0, 5], \dim(\mu_7) = 16, \deg(\mu_7) = 9;$$

$$\mu_8 = [0, 0, 0, 0, -6] + [0, 0, 0, 0, 6], \dim(\mu_8) = 2, \deg(\mu_8) = 10.$$

- D=8+2, n=1

$$\mu_0 = [0, 1, 0, 0, 0], \dim(\mu_0) = 28, \deg(\mu_0) = 4;$$

$$\mu_1 = [0, 0, 0, 1, -1] + [0, 0, 1, 0, 1] + [1, 0, 0, 1, 1] + [1, 0, 1, 0, -1], \dim(\mu_1) = 128, \deg(\mu_1) = 5;$$

$$\begin{aligned} \mu_2 = & [0, 0, 0, 0, -2] + [0, 0, 0, 0, 2] + [0, 0, 0, 2, 2] + [0, 0, 1, 1, 0] + [0, 0, 2, 0, -2] + \\ & + 2 \times [1, 0, 0, 0, 0] + [2, 0, 0, 0, -2] + [2, 0, 0, 0, 2], \dim(\mu_2) = 214, \deg(\mu_2) = 6; \end{aligned}$$

$$\mu_3 = \mu_3' + \mu_3'',$$

$$\mu_3' = [0, 0, 0, 1, 1] + [0, 0, 1, 0, -1] + [1, 0, 0, 1, 3] + [1, 0, 1, 0, -3], \dim(\mu_3') = 128, \deg(\mu_3') = 7;$$

$$\mu_3'' = [0, 0, 0, 0, 0] + [0, 0, 0, 2, 0] + [0, 0, 2, 0, 0] + [2, 0, 0, 0, 0], \dim(\mu_3'') = 106, \deg(\mu_3'') = 8;$$

$$\mu_4 = \mu_4' + \mu_4'',$$

$$\mu_4' = [0, 1, 0, 0, -4] + [0, 1, 0, 0, 4], \dim(\mu_4') = 56, \deg(\mu_4') = 8;$$

$$\mu_4'' = [0, 0, 0, 1, -1] + [0, 0, 1, 0, 1] + [1, 0, 0, 1, 1] + [1, 0, 1, 0, -1], \dim(\mu_4'') = 128, \deg(\mu_4'') = 9;$$

$$\mu_5 = \mu_5' + \mu_5'',$$

$$\mu_5' = [0, 0, 0, 1, -5] + [0, 0, 1, 0, 5], \dim(\mu_5') = 16, \deg(\mu_5') = 9;$$

$$\mu_5'' = [0, 1, 0, 0, -2] + [0, 1, 0, 0, 2] + [1, 0, 0, 0, 0], \dim(\mu_5'') = 64, \deg(\mu_5'') = 10;$$

$$\mu_6 = \mu_6' + \mu_6'',$$

$$\mu_6' = [0, 0, 0, 0, -6] + [0, 0, 0, 0, 6], \dim(\mu_6') = 2, \deg(\mu_6') = 10;$$

$$\mu_6'' = [0, 0, 0, 1, -3] + [0, 0, 1, 0, 3], \dim(\mu_6'') = 16, \deg(\mu_6'') = 11;$$

$$\mu_7 = [0, 0, 0, 0, -4] + [0, 0, 0, 0, 4], \dim(\mu_7) = 2, \deg(\mu_7) = 12.$$

- D=8+2, n=2

$$\mu_0 = [1, 0, 0, 0, 0], \dim(\mu_0) = 8, \deg(\mu_0) = 6;$$

$$\mu_1 = \mu_1' + \mu_1'',$$

$$\mu_1' = [0, 0, 0, 1, 1] + [0, 0, 1, 0, -1], \dim(\mu_1') = 16, \deg(\mu_1') = 7;$$

$$\mu_1'' = [2, 0, 0, 0, 0], \dim(\mu_1'') = 35, \deg(\mu_1'') = 8;$$

$$\mu_2 = \mu_2' + \mu_2'',$$

$$\mu_2' = [0, 0, 0, 0, 0], \dim(\mu_2') = 1, \deg(\mu_2') = 8;$$

$$\mu_2'' = [0, 0, 0, 1, -1] + [0, 0, 1, 0, 1] + [1, 0, 0, 1, 1] + [1, 0, 1, 0, -1], \dim(\mu_2'') = 128, \deg(\mu_2'') = 9;$$

$$\mu_3 = [0, 0, 0, 0, -2] + [0, 0, 0, 0, 2] + [0, 0, 1, 1, 0] + [0, 1, 0, 0, -2] + [0, 1, 0, 0, 2] + \\ + 2 \times [1, 0, 0, 0, 0], \dim(\mu_3) = 130, \deg(\mu_3) = 10;$$

$$\mu_4 = \mu_4' + \mu_4'',$$

$$\mu_4' = [0, 0, 0, 1, -3] + [0, 0, 0, 1, 1] + [0, 0, 1, 0, -1] + [0, 0, 1, 0, 3], \dim(\mu_4') = 32, \deg(\mu_4') = 11;$$

$$\mu_4'' = [0, 1, 0, 0, 0], \dim(\mu_4'') = 28, \deg(\mu_4'') = 12;$$

$$\mu_5 = \mu_5' + \mu_5'',$$

$$\mu_5' = [0, 0, 0, 0, -4] + [0, 0, 0, 0, 4], \dim(\mu_5') = 2, \deg(\mu_5') = 12;$$

$$\mu_5'' = [0, 0, 0, 1, -1] + [0, 0, 1, 0, 1], \dim(\mu_5'') = 16, \deg(\mu_5'') = 13;$$

$$\mu_6 = [0, 0, 0, 0, -2] + [0, 0, 0, 0, 2], \dim(\mu_6) = 2, \deg(\mu_6) = 14.$$

- D=8+2, n=3

$$\mu_0 = [0, 0, 0, 0, 0], \dim(\mu_0) = 1, \deg(\mu_0) = 8;$$

$$\mu_1 = [0, 0, 0, 0, -2] + [0, 0, 0, 0, 2] + [1, 0, 0, 0, 0], \dim(\mu_1) = 10, \deg(\mu_1) = 10;$$

$$\mu_2 = [0, 0, 0, 1, 1] + [0, 0, 1, 0, -1], \dim(\mu_2) = 16, \deg(\mu_2) = 11;$$

$$\mu_3 = [0, 0, 0, 1, -1] + [0, 0, 1, 0, 1], \dim(\mu_3) = 16, \deg(\mu_3) = 13;$$

$$\mu_4 = [0, 0, 0, 0, -2] + [0, 0, 0, 0, 2] + [1, 0, 0, 0, 0], \dim(\mu_4) = 10, \deg(\mu_4) = 14;$$

$$\mu_5 = [0, 0, 0, 0, 0], \dim(\mu_5) = 1, \deg(\mu_5) = 16.$$

- D=7+3, n=0

$$\mu_0 = [0, 0, 0, 0], \dim(\mu_0) = 1, \deg(\mu_0) = 0;$$

$$\mu_1 = [1, 0, 0, 0], \dim(\mu_1) = 7, \deg(\mu_1) = 2;$$

$$\mu_2 = [0, 1, 0, 0] + [1, 0, 0, 0], \dim(\mu_2) = 28, \deg(\mu_2) = 4;$$

$$\mu_3 = \mu_3' + \mu_3'',$$

$$\mu_3' = [0, 0, 1, 1], \dim(\mu_3') = 16, \deg(\mu_3') = 5;$$

$$\mu_3'' = [0, 0, 0, 0] + [0, 0, 2, 0] + [2, 0, 0, 0], \dim(\mu_3'') = 63, \deg(\mu_3'') = 6;$$

$$\mu_4 = \mu_4' + \mu_4'',$$

$$\mu_4' = [0, 0, 0, 2], \dim(\mu_4') = 3, \deg(\mu_4') = 6;$$

$$\mu_4'' = [0, 0, 1, 1] + [1, 0, 1, 1], \dim(\mu_4'') = 112, \deg(\mu_4'') = 7;$$

$$\mu_5 = [0, 0, 0, 0] + [0, 1, 0, 2] + [1, 0, 0, 2], \dim(\mu_5) = 85, \deg(\mu_5) = 8;$$

$$\mu_6 = [0, 0, 1, 3], \dim(\mu_6) = 32, \deg(\mu_6) = 9;$$

$$\mu_7 = [0, 0, 0, 4], \dim(\mu_7) = 5, \deg(\mu_7) = 10.$$

- D=7+3, n=1

$$\mu_0 = [1, 0, 0, 0], \dim(\mu_0) = 7, \deg(\mu_0) = 4;$$

$$\mu_1 = \mu_1' + \mu_1'',$$

$$\mu_1' = [0, 0, 1, 1], \dim(\mu_1') = 16, \deg(\mu_1') = 5;$$

$$\mu_1'' = [2, 0, 0, 0], \dim(\mu_1'') = 27, \deg(\mu_1'') = 6;$$

$$\mu_2 = \mu_2' + \mu_2'',$$

$$\mu_2' = [0, 0, 0, 2], \dim(\mu_2') = 3, \deg(\mu_2') = 6;$$

$$\mu_2'' = [0, 0, 1, 1] + [1, 0, 1, 1], \dim(\mu_2'') = 112, \deg(\mu_2'') = 7;$$

$$\mu_3 = [0, 0, 0, 0] + [0, 0, 0, 2] + [0, 0, 2, 0] + [0, 1, 0, 2] + [1, 0, 0, 0] + [1, 0, 0, 2],$$

$$\dim(\mu_3) = 130, \deg(\mu_3) = 8;$$

$$\mu_4 = \mu_4' + \mu_4'',$$

$$\mu_4' = [0, 0, 1, 1] + [0, 0, 1, 3], \dim(\mu_4') = 48, \deg(\mu_4') = 9;$$

$$\mu_4'' = [0, 1, 0, 0], \dim(\mu_4'') = 21, \deg(\mu_4'') = 10;$$

$$\mu_5 = \mu_5' + \mu_5'',$$

$$\mu_5' = [0, 0, 0, 4], \dim(\mu_5') = 5, \deg(\mu_5') = 10;$$

$$\mu_5'' = [0, 0, 1, 1], \dim(\mu_5'') = 16, \deg(\mu_5'') = 11;$$

$$\mu_6 = [0, 0, 0, 2], \dim(\mu_6) = 3, \deg(\mu_6) = 12.$$

- D=7+3, n=2

$$\mu_0 = [0, 0, 0, 0], \dim(\mu_0) = 1, \deg(\mu_0) = 6;$$

$$\mu_1 = [0, 0, 0, 2] + [1, 0, 0, 0], \dim(\mu_1) = 10, \deg(\mu_1) = 8;$$

$$\mu_2 = [0, 0, 1, 1], \dim(\mu_2) = 16, \deg(\mu_2) = 9;$$

$$\mu_3 = [0, 0, 1, 1], \dim(\mu_3) = 16, \deg(\mu_3) = 11;$$

$$\mu_4 = [0, 0, 0, 2] + [1, 0, 0, 0], \dim(\mu_4) = 10, \deg(\mu_4) = 12;$$

$$\mu_5 = [0, 0, 0, 0], \dim(\mu_5) = 1, \deg(\mu_5) = 14.$$

- D=6+4, n=0

$$\mu_0 = [0, 0, 0, 0, 0], \dim(\mu_0) = 1, \deg(\mu_0) = 0;$$

$$\mu_1 = [0, 1, 0, 0, 0], \dim(\mu_1) = 6, \deg(\mu_1) = 2;$$

$$\mu_2 = [0, 0, 0, 0, 0] + [1, 0, 1, 0, 0], \dim(\mu_2) = 16, \deg(\mu_2) = 4;$$

$$\mu_3 = [0, 0, 0, 1, 1] + [0, 0, 2, 0, 0] + [0, 1, 0, 0, 0] + [2, 0, 0, 0, 0], \dim(\mu_3) = 30, \deg(\mu_3) = 6;$$

$$\mu_4 = \mu_4' + \mu_4'',$$

$$\mu_4' = [0, 0, 1, 1, 0] + [1, 0, 0, 0, 1], \dim(\mu_4') = 16, \deg(\mu_4') = 7;$$

$$\mu_4'' = [1, 0, 1, 0, 0], \dim(\mu_4'') = 15, \deg(\mu_4'') = 8;$$

$$\mu_5 = [0, 0, 1, 0, 1] + [1, 0, 0, 1, 0], \dim(\mu_5) = 16, \deg(\mu_5) = 9;$$

$$\mu_6 = [0, 0, 0, 1, 1], \dim(\mu_6) = 4, \deg(\mu_6) = 10.$$

- D=6+4, n=1

$$\begin{aligned}
\mu_0 &= [0, 0, 0, 0, 0], \dim(\mu_0) = 1, \deg(\mu_0) = 4; \\
\mu_1 &= [0, 0, 0, 1, 1] + [0, 1, 0, 0, 0], \dim(\mu_1) = 10, \deg(\mu_1) = 6; \\
\mu_2 &= [0, 0, 1, 1, 0] + [1, 0, 0, 0, 1], \dim(\mu_2) = 16, \deg(\mu_2) = 7; \\
\mu_3 &= [0, 0, 1, 0, 1] + [1, 0, 0, 1, 0], \dim(\mu_3) = 16, \deg(\mu_3) = 9; \\
\mu_4 &= [0, 0, 0, 1, 1] + [0, 1, 0, 0, 0], \dim(\mu_4) = 10, \deg(\mu_4) = 10; \\
\mu_5 &= [0, 0, 0, 0, 0], \dim(\mu_5) = 1, \deg(\mu_5) = 12.
\end{aligned}$$

- D=5+5, n=0

$$\begin{aligned}
\mu_0 &= [0, 0, 0, 0], \dim(\mu_0) = 1, \deg(\mu_0) = 0; \\
\mu_1 &= [1, 0, 0, 0], \dim(\mu_1) = 5, \deg(\mu_1) = 2; \\
\mu_2 &= [0, 2, 0, 0], \dim(\mu_2) = 10, \deg(\mu_2) = 4; \\
\mu_3 &= [0, 2, 0, 0], \dim(\mu_3) = 10, \deg(\mu_3) = 6; \\
\mu_4 &= [1, 0, 0, 0], \dim(\mu_4) = 5, \deg(\mu_4) = 8; \\
\mu_5 &= [0, 0, 0, 0], \dim(\mu_5) = 1, \deg(\mu_5) = 10.
\end{aligned}$$

- D=4+6, n=0

$$\begin{aligned}
\mu_0 &= [0, 0, 0, 0, 0], \dim(\mu_0) = 1, \deg(\mu_0) = 0; \\
\mu_1 &= [1, 1, 0, 0, 0], \dim(\mu_1) = 4, \deg(\mu_1) = 2; \\
\mu_2 &= [0, 2, 0, 0, 0] + [2, 0, 0, 0, 0], \dim(\mu_2) = 6, \deg(\mu_2) = 4; \\
\mu_3 &= [1, 1, 0, 0, 0], \dim(\mu_3) = 4, \deg(\mu_3) = 6; \\
\mu_4 &= [0, 0, 0, 0, 0], \dim(\mu_4) = 1, \deg(\mu_4) = 8.
\end{aligned}$$



- D=3+7, n=0

$$\mu_0 = [0, 0, 0, 0], \dim(\mu_0) = 1, \deg(\mu_0) = 0;$$

$$\mu_1 = [2, 0, 0, 0], \dim(\mu_1) = 3, \deg(\mu_1) = 2;$$

$$\mu_2 = [2, 0, 0, 0], \dim(\mu_2) = 3, \deg(\mu_2) = 4;$$

$$\mu_3 = [0, 0, 0, 0], \dim(\mu_3) = 1, \deg(\mu_3) = 6.$$

- D=2+8, n=0

$$\mu_0 = [0, 0, 0, 0, 0], \dim(\mu_0) = 1, \deg(\mu_0) = 0;$$

$$\mu_1 = [0, 0, 0, 0, -2] + [0, 0, 0, 0, 2], \dim(\mu_1) = 2, \deg(\mu_1) = 2;$$

$$\mu_2 = [0, 0, 0, 0, 0], \dim(\mu_2) = 1, \deg(\mu_2) = 4.$$

- D=1+9, n=0

$$\mu_0 = [0, 0, 0, 0], \dim(\mu_0) = 1, \deg(\mu_0) = 0;$$

$$\mu_1 = [0, 0, 0, 0], \dim(\mu_1) = 1, \deg(\mu_1) = 2.$$

## B Computer calculations

We will describe here the computer programs used in the calculations.<sup>9</sup>

1. We calculate the differential  $d : V \rightarrow S \otimes S$  (Eq. 2) using Gamma [15].
2. We use Macaulay2 [9] to calculate the Poincare (Hilbert) series  $P_n(\tau) = \sum_k \dim^n M_k \tau^k$  of  $R$ -module  ${}^n M = \sum_k H^{k,n}$ . Here  $R = \mathbb{C}[t^1, \dots, t^\alpha, \dots] = \sum_m \text{Sym}^m S$ . We calculate generators of this module and generators of free resolution

$$\cdots \rightarrow {}^n M_i \rightarrow \cdots \rightarrow {}^n M_1 \rightarrow {}^n M_0 \rightarrow {}^n M \rightarrow 0$$

where  ${}^n M_i = {}^n \mu_i \otimes R$ .

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<sup>9</sup>The detailed codes are provided here: <http://lifshitz.ucdavis.edu/~rxu/code/cohom/>

Input:

Coefficients  $\Gamma_{\alpha\beta}^m$  in the differential, the number of Greek indices ( $\dim S$ ),  
the number of Roman indices ( $\dim V$ ).

Output:

Poincare (Hilbert) series,

number of generators of  ${}^nM$  and the number of them,

number of generators of  ${}^n\mu_i$  having given degree.

3. Using LiE, we decompose  $\text{Sym}^m S \otimes \wedge^k V$  into irreducible representation of *Aut*. Applying principle of maximal propagation and resolving the ambiguities from the information about Poincare series we obtain decomposition of cohomology into irreducible representation for  $k \geq 20$ .
4. We make a conjecture of the decomposition of  $H^{k,n}$  into irreducible representation for arbitrary  $k$  using the information from the step 3. We prove that our conjecture gives the right Poincare series using Weyl dimension formula.<sup>10</sup>
5. We make a conjecture about cohomology generators using the information about their numbers and dimension from Macaulay2 [9] and the information from the steps 3 and 4. We prove that our formulas give cocycles using Gamma [15].
6. We use the formula Eq. 105 to get the decomposition of generators of free resolution into irreducible representation.

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<sup>10</sup>The Mathematica code for 10D case is provided here:  
<http://lifshitz.ucdavis.edu/~rxu/code/cohom/dim10dredux.nb>

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# Homology of Lie algebra of supersymmetries and of super Poincare Lie algebra\*

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**Abstract** We study the homology and cohomology groups of super Lie algebra of supersymmetries and of super Poincare Lie algebra. We give complete answers for (non-extended) supersymmetry in all dimensions  $\leq 11$ . For dimensions  $D = 10, 11$  we describe also the cohomology of reduction of supersymmetry Lie algebra to lower dimensions. Our methods can be applied to extended supersymmetry algebra.

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# 1 Introduction

In present paper we will analyze homology and cohomology groups of the super Lie algebra of supersymmetries and of super Poincare Lie algebra. We came to this problem studying supersymmetric deformations of maximally supersymmetric gauge theories [14]; however, this problem arises also in different situations, in particular, in supergravity [2]. In low dimensions it was studied in [6]. The cohomology of supersymmetry Lie algebra appeared in the analysis of supersymmetric invariants in [5] (it was denoted there by the symbol  $H_t^{p,q}$ ). The cohomology groups calculated below appear also in pure spinor formalism of ten-dimensional supersymmetric gauge theory in [4]. Some of results of present paper were derived by more elementary methods in our previous paper [15].

Let us recall the definition of Lie algebra cohomology. We start with super Lie algebra  $\mathcal{G}$  with generators  $e_A$  and structure constants  $f_{AB}^K$ . We introduce ghost variables  $C^A$  with parity opposite to the parity of generators  $e_A$  and consider the algebra  $E$  of polynomial functions of these variables. (In more invariant way we can say that  $E$  consists of polynomial functions on linear superspace  $\Pi\mathcal{G}$ . The algebra  $E$  is graded by the degree of polynomial. We define a derivation  $d$  on  $E$  by the formula  $d = \frac{1}{2}f_{AB}^K C^A C^B \frac{\partial}{\partial C^K}$ .

This operator is a differential (i.e. it changes the parity and obeys  $d^2 = 0$ .) We define the cohomology of  $\mathcal{G}$  using this differential:

$$H^\bullet(\mathcal{G}) = \text{Ker } d / \text{Im } d.$$

The definition of homology of  $\mathcal{G}$  is dual to the definition of cohomology: instead of  $E$  we consider its dual space  $E^*$  that can be considered as the space of functions of dual ghost variables  $c_A$ ; the differential  $\partial$  on  $E^*$  is defined as an operator adjoint to  $d$ . The homology  $H_\bullet(\mathcal{G})$  is dual to the cohomology  $H^\bullet(\mathcal{G})$ . We will work with cohomology, but our results can be interpreted in the language of homology.

Notice that we can multiply cohomology classes, i.e.  $H^\bullet(\mathcal{G})$  is an algebra.

The group  $\text{Aut}(\mathcal{G})$  of automorphisms of  $\mathcal{G}$  acts on  $E$  and commutes with

the differential, therefore it acts also on homology and cohomology. We will be interested in this action. In other words we calculate cohomology as representation of this group or as a representation of its Lie algebra  $aut(\mathcal{G})$  (as an  $aut(\mathcal{G})$ -module). For every graded module  $E$  we can define its Euler characteristic  $\chi(E)$  as a virtual module  $\sum (-1)^k E_k$  (as an alternating sum of its graded components in the sense of K-theory). Euler characteristic of graded differential module coincides with Euler characteristic of its homology. This allows us to calculate the Euler characteristic of Lie algebra cohomology as virtual representation (virtual  $aut(\mathcal{G})$ -module) .<sup>1</sup> If the cohomology does not vanish only in one degree the Euler characteristic gives a complete answer for cohomology.

The super Lie algebra of supersymmetries has odd generators  $e_\alpha$  and even generators  $P_m$  ; the only non-trivial commutation relation is

$$[e_\alpha, e_\beta]_+ = \Gamma_{\alpha\beta}^m P_m.$$

The coefficients in this relation are expressed in terms of Dirac Gamma matrices (see e.g. [7] for mathematical introduction). The space  $E$  used in the definition of cohomology (cochain complex) consists here of polynomial functions of even ghost variables  $t^\alpha$  and odd ghost variables  $c^m$ ; the differential has the form

$$d = \frac{1}{2} \Gamma_{\alpha\beta}^m t^\alpha t^\beta \frac{\partial}{\partial c^m}. \quad (2)$$

The space  $E$  is double-graded (one can consider the degree with respect to  $t^\alpha$  and the degree with respect to  $c^m$ ). In more invariant form we can say that<sup>2</sup>

$$E = \bigoplus \text{Sym}^m S \otimes \Lambda^n V$$

where  $S$  stands for spinorial representation of orthogonal group,  $V$  denotes vector representation of this group and Gamma-matrices specify an intertwiner  $V \rightarrow \text{Sym}^2 S$ . The differential  $d$  maps  $\text{Sym}^m S \otimes \Lambda^n V$  into  $\text{Sym}^{m+2} S \otimes \Lambda^{n-1} V$ .

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<sup>1</sup>Instead of virtual modules we can talk about virtual representations of  $Aut(\mathcal{G})$  (elements of representation ring). If the group  $Aut(\mathcal{G})$  is compact the representation ring can be identified with the ring of characters.

<sup>2</sup>We use the notation  $\text{Sym}^m$  for symmetric tensor power and the notation  $\Lambda^n$  for exterior power

The above description can be applied to any dimension and to any signature of the metric used in the definition of orthogonal group, however, spinorial representation is dimension-dependent.<sup>3</sup> The group  $SO(n)$  can be considered as a (subgroup) of the group of automorphisms of supersymmetry Lie algebra and therefore it acts on its cohomology. The action of  $SO(n)$  is two-valued, hence it would be more precise to talk about action of its two-sheeted covering  $Spin(n)$  or about action of its Lie algebra  $\mathfrak{so}(n)$ . We will work with complex representations and complex Lie algebras; this does not change the cohomology

We will consider also homology and cohomology of reduced Lie algebra of supersymmetries (or more precisely the Lie algebra of supersymmetries in dimension  $n$  reduced to the dimension  $d$ ). This algebra has the same odd generators  $e_\alpha$  as the Lie algebra of supersymmetries in dimension  $n$ , but only  $d \leq \dim V$  even generators  $P_1, \dots, P_d$ ; the commutation relations are the same as in unreduced algebra. In this case the cohomology is a representation of  $Spin(d) \times Spin(n-d)$ .

The double grading on  $E$  induces double grading on cohomology. However, instead of the degrees  $m$  and  $n$  it is more convenient to use the degrees  $k = m + 2n$  and  $n$  because the differential preserves  $k$  and therefore the problem of calculation of cohomology can be solved for every  $k$  separately. It is important to notice that the differential commutes with multiplication by a polynomial in  $t^\alpha$ , therefore the cohomology is a module over the polynomial ring  $\mathbb{C}[t^1, \dots, t^\alpha, \dots]$ . (Moreover, it is an algebra over this ring.) The cohomology is infinite-dimensional as a vector space, but it has a finite number of generators as a  $\mathbb{C}[t^1, \dots, t^\alpha, \dots]$ -module (this follows from the fact that the polynomial ring is noetherian). One of the problems we would like to solve is the description of these generators. If cohomology classes of cocycles  $z_1, \dots, z_N$  generate the cohomology then every cohomology class can be represented by a cocycle of the

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<sup>3</sup>Recall that orthogonal group  $SO(2n)$  has two irreducible two-valued complex representations called semi-spin representations (left spinors and right spinors), the orthogonal group  $SO(2n+1)$  has one irreducible two-valued complex spin representation. One says that a real representation is spinorial if after extension of scalars to  $\mathbb{C}$  it becomes a sum of spin or semi-spin representations. (We follow the terminology of [7].)



form  $p_1 z_1 + \dots + p_N z_N$  where  $p_1, \dots, p_N$  belong to  $\mathbb{C}[t^1, \dots, t^\alpha, \dots]$ .

Notice that the cohomology of Lie algebra of supersymmetries can be interpreted as homology of Koszul complex <sup>4</sup> corresponding to a sequence of functions  $f^m(t) = \frac{1}{2} \Gamma_{\alpha\beta}^m t^\alpha t^\beta$ . This allows us to use software [10] to calculate the dimensions of cohomology groups. However, we are interested in more complicated problem- in the description of decomposition of cohomology groups in direct sum of irreducible representations of the group of automorphisms *Aut* or its Lie algebra *aut*. We use [9] for such calculations.

The paper is organized as follows. We start with the description of cohomology of Lie algebra of supersymmetries in dimension 10 (Sec.2) and in dimension 11 (Sec.3). In the next sections we describe cohomology of dimensional reductions of ten-dimensional algebra of supersymmetries (Sec.4) and of eleven-dimensional supersymmetries (Sec.5). Section 6 contains the results about Lie algebras of supersymmetries in dimensions  $\leq 9$ . Section 7 is devoted to the explanation of methods we are using. Section 8 is devoted to cohomology of super Poincare Lie algebra. The paper contains two appendices that will be omitted in printed version. In Appendix A we describe the decomposition of free resolution in direct sum of representations of the automorphism group. Appendix B gives more detail about our calculations.

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## 2 D=10

We will start with ten-dimensional case; in this case the spinorial representation in the definition of Lie algebra of supersymmetries should be considered as one of two irreducible 16-dimensional representations of  $\text{Spin}(10)$  (in Minkowski space the spinors are Majorana-Weyl spinors). The Lie algebra of automorphisms *aut* is  $\mathfrak{so}(10)$ .

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<sup>4</sup>To define the homology of Koszul complex corresponding to a sequence of functions  $f^1(t), \dots, f^n(t)$  one considers the differential  $d = f^m(t) \frac{\partial}{\partial \xi_m}$  where  $\xi_1, \dots, \xi_n$  are odd variables.

We will describe representations of the Lie algebra  $\mathfrak{so}(10)$  in the cohomology of the Lie algebra of supersymmetries in ten dimensions. As usual the representations are labeled by coordinates of their highest weight (see e.g. [17] for details). The vector representation  $V$  has the highest weight  $[1, 0, 0, 0, 0]$ , the irreducible spinor representations have highest weights  $[0, 0, 0, 0, 1], [0, 0, 0, 1, 0]$ ; we assume that the highest weight of  $S$  is  $[0, 0, 0, 0, 1]$ . The description of graded component of cohomology group with gradings  $k = m + 2n$  and  $n$  is given by the formulas for  $H^{k,n}$  (for  $n \geq 6$ ,  $H^{k,n}$  vanishes)

$$H^{k,0} = [0, 0, 0, 0, k] \quad (3)$$

$$H^{k,1} = [0, 0, 0, 1, k-3] \quad (4)$$

$$H^{k,2} = [0, 0, 1, 0, k-6] \quad (5)$$

$$H^{k,3} = [0, 1, 0, 0, k-8] \quad (6)$$

$$H^{k,4} = [1, 0, 0, 0, k-10] \quad (7)$$

$$H^{k,5} = [0, 0, 0, 0, k-12] \quad (8)$$

The only special case is when  $k = 4$ , there is one additional term, a scalar, for  $H^{4,1}$ .

$$H^{4,1} = [0, 0, 0, 0, 0] \oplus [0, 0, 0, 1, 1] \quad (9)$$

The  $\text{SO}(10)$ -invariant part is in  $H^{0,0}$ ,  $H^{12,5}$ , and  $H^{4,1}$ .

The dimensions of these cohomology groups are encoded in series  $P_n(\tau) =$

$\sum_k \dim H^{k,n} \tau^k$  (Poincare series) that can be calculated by means of [10]:

$$P_0(\tau) = \frac{\tau^3 + 5\tau^2 + 5\tau + 1}{(1 - \tau)^{11}}, \quad (10)$$

$$P_1(\tau) = \frac{(16\tau^3 + 35\tau^4 - \tau^5 + 55\tau^6 - 165\tau^7 + 330\tau^8 - 462\tau^9 + 462\tau^{10} - 330\tau^{11} + 165\tau^{12} - 55\tau^{13} + 11\tau^{14} - \tau^{15})}{(1 - \tau)^{11}}, \quad (11)$$

$$P_2(\tau) = \frac{(120\tau^6 - 120\tau^7 + 330\tau^8 - 462\tau^9 + 462\tau^{10} - 330\tau^{11} + 165\tau^{12} - 55\tau^{13} + 11\tau^{14} - \tau^{15})}{(1 - \tau)^{11}}, \quad (12)$$

$$P_3(\tau) = \frac{45\tau^8 + 65\tau^9 + 11\tau^{10} - \tau^{11}}{(1 - \tau)^{11}}, \quad (13)$$

$$P_4(\tau) = \frac{10\tau^{10} + 34\tau^{11} + 16\tau^{12}}{(1 - \tau)^{11}}, \quad (14)$$

$$P_5(\tau) = \frac{\tau^{12} + 5\tau^{13} + 5\tau^{14} + \tau^{15}}{(1 - \tau)^{11}} \quad (15)$$

The cohomology regarded as  $\mathbb{C}[t^1, \dots, t^\alpha, \dots]$ -module is generated by the scalar considered as an element of  $H^{0,0}$  and by

$$\begin{aligned} [t^\alpha c_m \Gamma_{\alpha\beta}^m] &\in H^{3,1}, \\ [t^\alpha t^\beta c_m c_n \Gamma_{\alpha\beta}^{mnklr}] &\in H^{6,2}, \\ [t^\alpha t^\beta c_m c_n c_k \Gamma_{\alpha\beta}^{mnklr}] &\in H^{8,3}, \\ [t^\alpha t^\beta c_m c_n c_k c_l \Gamma_{\alpha\beta}^{mnklr}] &\in H^{10,4}, \\ [t^\alpha t^\beta c_m c_n c_k c_l c_r \Gamma_{\alpha\beta}^{mnklr}] &\in H^{12,5}. \end{aligned}$$

Here  $[a]$  denotes the cohomological class of cocycle  $a$ .

The GAMMA package [16] was used to verify that the expression above are cocycles .

### 3 D=11

Now we consider the eleven-dimensional case; in this case the spinorial representation in the definition of supersymmetry Lie algebra should be considered as one irreducible 32-dimensional spinor representations of Spin(11) (Dirac spinors). As usual we work with complex representations and complex Lie algebras. .

We will describe representations of  $Aut(\mathcal{G}) = \mathfrak{so}(11)$  in the the cohomology of the Lie algebra of supersymmetries. As usual the representations are labeled by their highest weight. The vector representation  $V$  has the highest weight  $[1, 0, 0, 0, 0]$ , the irreducible spinor representations have highest weights  $[0, 0, 0, 0, 1]$ . The description of graded component of cohomology group with gradings  $k = m + 2n$  and  $n$  is given by the formulas for  $H^{k,n}$  (for  $n \geq 3$ ,  $H^{k,n}$  vanishes)

$$H^{k,0} = \bigoplus_{i=0}^{[k/2]} [0, i, 0, 0, k - 2i] \quad (16)$$

$$H^{k,1} = \bigoplus_{i=0}^{[(k-4)/2]} [1, i, 0, 0, k - 4 - 2i] \quad (17)$$

$$H^{k,2} = \bigoplus_{i=0}^{[(k-6)/2]} [0, i, 0, 0, k - 6 - 2i] \quad (18)$$

The  $SO(11)$ -invariant part is in  $H^{0,0}$  and  $H^{6,2}$ .

The dimensions of these cohomology groups are encoded in Poincare series:

$$P_0(\tau) = A(\tau) \quad (19)$$

$$P_1(\tau) = \frac{\tau^4(11 + 67\tau + 142\tau^2 + 142\tau^3 + 67\tau^4 + 11\tau^5)}{(1 - \tau)^{23}}, \quad (20)$$

$$P_2(\tau) = A(\tau)\tau^6 \quad (21)$$

where

$$A(\tau) = \frac{1 + 9\tau + 34\tau^2 + 66\tau^3 + 66\tau^4 + 34\tau^5 + 9\tau^6 + \tau^7}{(1 - \tau)^{23}} \quad (22)$$

The cohomology regarded as  $\mathbb{C}[t^1, \dots, t^\alpha, \dots]$ -module is generated by the scalar considered as an element of  $H^{0,0}$  and

$$\begin{aligned} [t^\alpha t^\beta c_m \Gamma_{\alpha\beta}^{mn}] &\in H^{4,1}, \\ [t^\alpha t^\beta c_m c_n \Gamma_{\alpha\beta}^{mn}] &\in H^{6,2}, \end{aligned}$$

## 4 Dimensional reduction from $D = 10$

Let us consider dimensional reductions of ten-dimensional Lie algebra of supersymmetries. The reduction of  $\mathfrak{su}\mathfrak{sh}_{10}$  to  $r$  dimensions has 16 odd generators (supersymmetries) and  $r$  even generators (here  $0 \leq r \leq 10$ ). Corresponding

differential has the form (2) where  $\Gamma_{\alpha\beta}^m$  are ten-dimensional Dirac matrices, Greek indices take 16 values as in unreduced case, but Roman indices take only  $d$  values. The differential commutes with (two-valued) action of the group  $\text{SO}(r) \times \text{SO}(10-r)$ , therefore this group acts on cohomology. The cohomology can be regarded as a module over  $\mathbb{C}[t^1, \dots, t^{16}]$ . Again cohomology is double graded; we use notation  $H^{k,n}$  for the component having degree  $m = k - 2n$  with respect to  $t$  and the degree  $n$  with respect to  $c$ . The symbol  $P_n(\tau)$  stands for the generating function  $P_n(\tau) = \sum_k \dim H^{k,n} \tau^k$  (for Poincare series). We calculate the cohomology as a representation of Lie algebra  $\mathfrak{so}(r) \times \mathfrak{so}(10-r)$  and describe elements that generate it as a  $\mathbb{C}[t^1, \dots, t^{16}]$ -module. (We characterize the representation writing Dynkin labels of the first factor, then Dynkin labels of second factor.)

- $r = 9$ ,

$$H^{k,0} = [0, 0, 0, k], k \neq 2 \quad (23)$$

$$H^{k,1} = [0, 0, 1, k-4] \quad (24)$$

$$H^{k,2} = [0, 1, 0, k-6] \quad (25)$$

$$H^{k,3} = [1, 0, 0, k-8] \quad (26)$$

$$H^{k,4} = [0, 0, 0, k-10] \quad (27)$$

when  $k = 2$ ,

$$H^{2,0} = [0, 0, 0, 0] \oplus [0, 0, 0, 2] \quad (28)$$

Groups  $H^{k,n}$  with  $n \geq 5$  vanish. The  $\text{SO}(9)$ -invariant part is in  $H^{0,0}$ ,  $H^{10,4}$ , and  $H^{2,0}$ .

Generators

$$\begin{aligned} [t^\alpha t^\beta c_m \Gamma_{\alpha\beta}^{mnkl}] &\in H^{4,1}, \\ [t^\alpha t^\beta c_m c_n \Gamma_{\alpha\beta}^{mnkl}] &\in H^{6,2}, \\ [t^\alpha t^\beta c_m c_n c_k \Gamma_{\alpha\beta}^{mnkl}] &\in H^{8,3}, \\ [t^\alpha t^\beta c_m c_n c_k c_l \Gamma_{\alpha\beta}^{mnkl}] &\in H^{10,4}. \end{aligned}$$

Poincare series

$$P_0(\tau) = (\tau^{13} - 11\tau^{12} + 55\tau^{11} - 165\tau^{10} + 330\tau^9 - 462\tau^8 + 462\tau^7 - 330\tau^6 + 165\tau^5 - 55\tau^4 + 10\tau^3 - 6\tau^2 - 5\tau - 1)/(-1 + \tau)^{11} \quad (29)$$

$$P_1(\tau) = (84\tau^4 - 156\tau^5 + 330\tau^6 - 462\tau^7 + 462\tau^8 - 330\tau^9 + 165\tau^{10} - 55\tau^{11} + 11\tau^{12} - \tau^{13})/(1 - \tau)^{11}, \quad (30)$$

$$P_2(\tau) = \frac{36\tau^6 + 36\tau^7}{(1 - \tau)^{11}}, \quad (31)$$

$$P_3(\tau) = \frac{9\tau^8 + 29\tau^9 + 11\tau^{10} - \tau^{11}}{(1 - \tau)^{11}}, \quad (32)$$

$$P_4(\tau) = \frac{\tau^{10} + 5\tau^{11} + 5\tau^{12} + \tau^{13}}{(1 - \tau)^{11}} \quad (33)$$

- $r = 8, k > 0$ ,

$$H^{k,0} = \bigoplus_{i=1}^{k-1} [0, 0, k - i, i, k - 2i] \bigoplus_{i=0}^{[k/2]} [0, 0, k - 2i, 0, k] \bigoplus_{i=0}^{[k/2]} [0, 0, 0, k - 2i, -k], \quad (34)$$

$$H^{k,1} = \bigoplus_{i=0}^{k-4} [0, 1, k - 4 - i, i, k - 4 - 2i], \quad (35)$$

$$H^{k,2} = \bigoplus_{i=0}^{k-6} [1, 0, k - 6 - i, i, k - 6 - 2i], \quad (36)$$

$$H^{k,3} = \bigoplus_{i=0}^{k-8} [0, 0, k - 8 - i, i, k - 8 - 2i] \quad (37)$$

Groups  $H^{k,n}$  with  $n \geq 4$  vanish. The  $\text{SO}(8) \times \text{SO}(2)$ -invariant part is in  $H^{0,0}$ , and  $H^{8,3}$ .

Generators:

$$\begin{aligned} [t^\alpha t^\beta c_m \Gamma_{\alpha\beta}^{mnk}] &\in H^{4,1}, \\ [t^\alpha t^\beta c_m c_n \Gamma_{\alpha\beta}^{mnk}] &\in H^{6,2}, \\ [t^\alpha t^\beta c_m c_n c_k \Gamma_{\alpha\beta}^{mnk}] &\in H^{8,3}, \end{aligned}$$

Poincare series

$$P_0(\tau) = \frac{2\tau^5 - 6\tau^4 + 5\tau^3 - 7\tau^2 - 5\tau - 1}{(-1 + \tau)^{11}}, \quad (38)$$

$$P_1(\tau) = \frac{28\tau^4 + 12\tau^5 - 6\tau^6 + 2\tau^7}{(1 - \tau)^{11}}, \quad (39)$$

$$P_2(\tau) = \frac{8\tau^6 + 24\tau^7 + 6\tau^8 - 2\tau^9}{(1 - \tau)^{11}}, \quad (40)$$

$$P_3(\tau) = \frac{\tau^8 + 5\tau^9 + 5\tau^{10} + \tau^{11}}{(1 - \tau)^{11}} \quad (41)$$

- $r = 7$ ,

$$H^{k,0} = \bigoplus_{i=1}^{[k/2]} [0, i, k - 2i, k - 2i] \bigoplus_{i=0}^{[k/2]} [0, 0, k - 2i, k] \quad (42)$$

$$H^{k,1} = \bigoplus_{i=0}^{[(k-4)/2]} [1, i, k - 4 - 2i, k - 4 - 2i] \quad (43)$$

$$H^{k,2} = \bigoplus_{i=0}^{[(k-6)/2]} [0, i, k - 6 - 2i, k - 6 - 2i] \quad (44)$$

Groups  $H^{k,n}$  with  $n \geq 3$  vanish. The  $\text{SO}(7) \times \text{SO}(3)$ -invariant part is in  $H^{0,0}$ , and  $H^{6,2}$ .

Generators:

$$\begin{aligned} [t^\alpha t^\beta c_m \Gamma_{\alpha\beta}^{mn}] &\in H^{4,1}, \\ [t^\alpha t^\beta c_m c_n \Gamma_{\alpha\beta}^{mn}] &\in H^{6,2}, \end{aligned}$$

Poincare series

$$P_0(\tau) = \frac{5\tau^5 - 7\tau^4 + 8\tau^2 + 5\tau + 1}{(-1 + \tau)^{11}}, \quad (45)$$

$$P_1(\tau) = \frac{7\tau^4 + 19\tau^5 + \tau^6 - 3\tau^7}{(1 - \tau)^{11}}, \quad (46)$$

$$P_2(\tau) = \frac{\tau^6 + 5\tau^7 + 5\tau^8 + \tau^9}{(1 - \tau)^{11}} \quad (47)$$

- $r = 6$ ,

$$\begin{aligned} H^{k,0} &= \bigoplus_{i=0}^{[k/2]} \bigoplus_{j=0}^{k-2i} [j, i, k - j - 2i, j, k - j - 2i] \\ &\quad \bigoplus_{i=1}^{k-1} \bigoplus_{j=\max\{0, 2i-k\}}^{i-1} [j, 0, k - 2i + j, i, k - i] \end{aligned} \quad (48)$$

$$H^{k,1} = \bigoplus_{i=0}^{[(k-4)/2]} \bigoplus_{j=0}^{k-4-2i} [j, i, k - 4 - j - 2i, j, k - 4 - j - 2i] \quad (49)$$

Groups  $H^{k,n}$  with  $n \geq 2$  vanish. The  $\text{SO}(6) \times \text{SO}(4)$ -invariant part is in  $H^{0,0}$ , and  $H^{4,1}$ .

Generators:

$$[t^\alpha t^\beta c_m \Gamma_{\alpha\beta}^m] \in H^{4,1} \quad (50)$$

Poincare series

$$P_0(\tau) = \frac{4\tau^5 + 4\tau^4 - 5\tau^3 - 9\tau^2 - 5\tau - 1}{(-1 + \tau)^{11}}, \quad (51)$$

$$P_1(\tau) = \frac{\tau^4 + 5\tau^5 + 5\tau^6 + \tau^7}{(1 - \tau)^{11}} \quad (52)$$

- $r = 5$ ,

$$H^{k,0} = \bigoplus_{i=1}^{[k/2]} \bigoplus_{j=0}^{i-1} [j, k-2i, i, k-2i] \bigoplus_{i=0}^{[k/2]} \bigoplus_{j=0}^{[(k-2i)/2]} [i, k-2i-2j, i, k-2i-2j] \quad (53)$$

Groups  $H^{k,n}$  with  $n \geq 1$  vanish. The  $\text{SO}(5) \times \text{SO}(5)$ -invariant part lies in  $H^{k,0}$  where  $k$  is even.

Poincare series

$$P_0(\tau) = \frac{(1 + \tau)^5}{(1 - \tau)^{11}} \quad (54)$$

- $r = 4$ ,

$$H^{k,0} = \bigoplus_{i=0}^{[k/2]} \bigoplus_{j=0}^{k-2i} (i+1) \times [j, k-2i-j, j, i, k-2i-j] \quad (55)$$

where the coefficient  $(i+1)$  is the multiplicity. Groups  $H^{k,n}$  with  $n \geq 1$  vanish. The  $\text{SO}(4) \times \text{SO}(6)$ -invariant part is in  $H^{0,0}$ .

Poincare series

$$P_0(\tau) = \frac{(1 + \tau)^4}{(1 - \tau)^{12}} \quad (56)$$

- $r = 3$ ,

$$H^{k,0} = \bigoplus_{i=0}^{[k/2]} \bigoplus_{j=0}^i [k-2i, j, i-j, k-2i] \quad (57)$$

Groups  $H^{k,n}$  with  $n \geq 1$  vanish. The  $\text{SO}(3) \times \text{SO}(7)$ -invariant part is in  $H^{0,0}$ .

Poincare series

$$P_0(\tau) = \frac{(1 + \tau)^3}{(1 - \tau)^{13}} \quad (58)$$



- $r = 2$ ,

$$H^{k,0} = \bigoplus_{i=0}^{[k/2]} \bigoplus_{j=0}^{k-2i} [i, 0, k-2i-j, j, k-2i-2j] \quad (59)$$

Groups  $H^{k,n}$  with  $n \geq 1$  vanish. The  $\text{SO}(2) \times \text{SO}(8)$ -invariant part is in  $H^{0,0}$ .

Poincare series

$$P_0(\tau) = \frac{(1+\tau)^2}{(1-\tau)^{14}} \quad (60)$$

- $r = 1$ ,

$$H^{k,0} = \bigoplus_{i=0}^{[k/2]} [i, 0, 0, k-2i] \quad (61)$$

Groups  $H^{k,n}$  with  $n \geq 1$  vanish. The  $\text{SO}(1) \times \text{SO}(9)$ -invariant part is in  $H^{0,0}$ .

Poincare series

$$P_0(\tau) = \frac{1+\tau}{(1-\tau)^{15}} \quad (62)$$

For  $r \leq 5$ , the cohomology is generated by the scalar 1.

## 5 Dimensional reduction from $D = 11$

Let us consider dimensional reductions of eleven-dimensional Lie algebra of supersymmetries. The reduction of  $\mathfrak{su}\mathfrak{m}_{11}$  to  $r$  dimensions has 32 odd generators (supersymmetries) and  $r$  even generators (here  $0 \leq r \leq 11$ ). Corresponding differential has the form (2) where  $\Gamma_{\alpha\beta}^m$  are eleven-dimensional Dirac matrices, Greek indices take 32 values as in unreduced case, but Roman indices take only  $d$  values. The differential commutes with action of the group  $\text{SO}(r) \times \text{SO}(11-r)$ , therefore this group acts on cohomology. The cohomology can be regarded as a module over  $\mathbb{C}[t^1, \dots, t^{32}]$ . Again cohomology is double graded; we use notation  $H^{k,n}$  for the component having degree  $m = k - 2n$  with respect to  $t$  and the degree  $n$  with respect to  $c$ . The symbol  $P_n(\tau)$  stands for the generating function  $P_n(\tau) = \sum_k \dim H^{k,n} \tau^k$  (for Poincare series). We calculate the cohomology as a representation of Lie algebra  $\mathfrak{so}(r) \times \mathfrak{so}(11-r)$  and describe elements that generate it as a  $\mathbb{C}[t^1, \dots, t^{32}]$ -module.

Let us start with calculation of Euler characteristic  $\chi(H^k)$  of cohomology  $H^k = \sum_n H^{k,n}$ . By general theorem this is a virtual  $\mathfrak{so}(r) \times \mathfrak{so}(11-r)$ -module

$$\sum_n (-1)^n \text{Sym}^{k-2n} S \otimes \Lambda^n V \quad (63)$$

where  $S$  and  $V$  are considered as  $\mathfrak{so}(r) \times \mathfrak{so}(11-r)$ -modules.

Cohomology for  $r = 10$  are given by the formula

$$\begin{aligned} H^{k,0} &= \bigoplus_{i=0}^{[k/2]} \bigoplus_{j=0}^{k-2i} \bigoplus_{l=0, l \neq 2}^{[\frac{k-2i-j}{2}]} [0, i, 0, j, k-2i-j-2l] \\ &\quad \bigoplus_{i=1}^{[k/2]} \bigoplus_{j=0}^{[\frac{k-2i}{2}]} \bigoplus_{l=0}^{k-2i-2j} [i, j, 0, l, k-2i-2j-l] \\ &\quad \bigoplus_{i=0}^{[\frac{k-4}{2}]} \bigoplus_{j=0}^{k-4-2i} [0, i, 0, j, k-4-2i-j], \end{aligned} \quad (64)$$

$$H^{k,1} = \bigoplus_{i=0}^{[\frac{k-4}{2}]} \bigoplus_{j=0}^i \bigoplus_{l=0}^{k-4-2i} [j, i-j, 0, l, k-4-2i-l] \quad (65)$$

$$(66)$$

Groups  $H^{k,n}$  with  $n \geq 2$  vanish. The  $\text{SO}(10)$ -invariant part is in  $H^{0,0}$  and  $H^{4,1}$ .

Generators

$$[t^\alpha t^\beta c_m \Gamma_{\alpha\beta}^m] \in H^{4,1} \quad (67)$$

Poincare series

$$P_0(\tau) = \frac{(1+\tau)^{10}}{(1-\tau)^{22}} + \tau^4 A(\tau), \quad (68)$$

$$P_1(\tau) = \tau^4 A(\tau) \quad (69)$$

where  $A(\tau)$  is the Poincare series given by Eq. 22.

For  $r \leq 9$  the groups  $H^{k,n}$  with  $n \geq 1$  vanish hence Euler characteristic (63) gives a complete description of cohomology.

Poincare series

$$P_0(\tau) = \frac{(1+\tau)^r}{(1-\tau)^{32-r}} \quad (70)$$

To find the  $\mathfrak{so}(r) \times \mathfrak{so}(11-r)$ -invariant part of  $H^{k,0}$  it is sufficient to solve this problem for Euler characteristic. The conjectural answers (obtained by means of computations for  $k < 19$ ) are listed below.

$$r = 1,$$

$$[0, 0, 0, 0, 0] \in H^{4k,0} \quad (71)$$

$$r = 2,$$

$$([k/2] + 1) \times [0, 0, 0, 0, 0] \in H^{2k,0} \quad (72)$$

$$r = 3,$$

$$(k + 1) \times [0, 0, 0, 0, 0] \in H^{4k,0} \quad (73)$$

$$r = 4,$$

$$\frac{(k+1)(k+2)}{2} \times [0, 0, 0, 0, 0] \in H^{4k,0} \quad (74)$$

$$r = 5,$$

$$\frac{([k/2] + 1)([k/2] + 2)([k/2] + 3)}{6} \times [0, 0, 0, 0, 0] \in H^{2k,0} \quad (75)$$

$$r = 6,$$

$$\frac{([k/2] + 1)([k/2] + 2)}{2} \times [0, 0, 0, 0, 0] \in H^{2k,0} \quad (76)$$

$$r = 7,$$

$$(k + 1) \times [0, 0, 0, 0, 0] \in H^{2k,0} \quad (77)$$

$$r = 8,$$

$$(2k + 1) \times [0, 0, 0, 0, 0] \in H^{2k,0} \quad (78)$$

$$r = 9,$$

$$2 \times [0, 0, 0, 0, 0] \in H^{4k,0}, \quad (79)$$

$$[0, 0, 0, 0, 0] \in H^{4k+2,0} \quad (80)$$

where  $i \times [a, b, c, d, e]$  denotes the representation  $[a, b, c, d, e]$  with multiplicity  $i$  and  $[a]$  stands for the integer part of  $a$ .

## 6 Other dimensions

In this section we consider in detail cohomology of Lie algebra of supersymmetries in dimensions  $< 10$ . Let us begin with some general discussion of supersymmetries in various dimensions (see [7] and [13] for more detail).

We will work with complex Lie algebras. Let us start with the description of the symmetric intertwiners  $\Gamma : S^* \otimes S^* \rightarrow V$  used in the construction of supersymmetry Lie algebra in various dimensions (notice that in the construction of differential we use dual intertwiners). Recall that in even dimensions we have two irreducible spinorial representations  $\mathfrak{s}_l$  and  $\mathfrak{s}_r$ , in odd dimensions we have one irreducible spinorial representation  $\mathfrak{s}$ .

- $\dim V = 8n$

In this case we have intertwiners  $\gamma_l : \mathfrak{s}_l \otimes \mathfrak{s}_r \rightarrow V$  and  $\gamma_r : \mathfrak{s}_r \otimes \mathfrak{s}_l \rightarrow V$ .

$$S \cong S^* = \mathfrak{s}_l + \mathfrak{s}_r, \quad \Gamma = \gamma_l + \gamma_r, \quad \dim S = 16^n.$$

Automorphism Lie algebra  $\text{aut} = \mathfrak{so}(8n) \times \mathfrak{so}(2)$ .

- $\dim V = 8n + 1$

In this case we have symmetric intertwiner  $\gamma : \mathfrak{s} \otimes \mathfrak{s} \rightarrow V$ .

$$S = S^* = \mathfrak{s}, \quad \Gamma = \gamma, \quad \dim S = 16^n.$$

Automorphism Lie algebra  $\text{aut} = \mathfrak{so}(8n + 1)$ .

- $\dim V = 8n + 2$

In this case we have symmetric intertwiners  $\gamma_l : \mathfrak{s}_l \otimes \mathfrak{s}_l \rightarrow V$  and  $\gamma_r : \mathfrak{s}_r \otimes \mathfrak{s}_r \rightarrow V$ .

There are two possible choices of  $S$  :

$$S = \mathfrak{s}_r, S^* = \mathfrak{s}_l, \Gamma = \gamma_l; \quad S = \mathfrak{s}_l, S^* = \mathfrak{s}_r, \Gamma = \gamma_r, \dim S = 16^n.$$

Automorphism Lie algebra  $\text{aut} = \mathfrak{so}(8n + 2)$ .

- $\dim V = 8n + 3$

In this case we have symmetric intertwiner  $\gamma : \mathfrak{s} \otimes \mathfrak{s} \rightarrow V$ .

$$S = S^* = \mathfrak{s}, \quad \Gamma = \gamma, \quad \dim S = 2 \times 16^n.$$

Automorphism Lie algebra  $\text{aut} = \mathfrak{so}(8n + 3)$ .

- $\dim V = 8n + 4$

In this case we have intertwiners  $\gamma_l : \mathfrak{s}_l \otimes \mathfrak{s}_r \rightarrow V$  and  $\gamma_r : \mathfrak{s}_r \otimes \mathfrak{s}_l \rightarrow V$ .

$$S \cong S^* = \mathfrak{s}_l + \mathfrak{s}_r, \quad \Gamma = \gamma_l + \gamma_r, \quad \dim S = 4 \times 16^n.$$

Automorphism Lie algebra  $\text{aut} = \mathfrak{so}(8n + 4) \times \mathfrak{so}(2)$ .

- $\dim V = 8n + 5$

The intertwiner  $\gamma : \mathfrak{s} \otimes \mathfrak{s} \rightarrow V$  is antisymmetric.

$$S \cong S^* = \mathfrak{s} \otimes W, \Gamma = \gamma \otimes \omega, \dim S = 8 \times 16^n.$$

Here and later  $W$  stand for two-dimensional linear space with a symplectic form  $\omega$ . Automorphism Lie algebra  $\text{aut} = \mathfrak{so}(8n + 5) \times \mathfrak{sl}(2)$ .

- $\dim V = 8n + 6$

In this case we have antisymmetric intertwiners  $\gamma_l : \mathfrak{s}_l \otimes \mathfrak{s}_l \rightarrow V$  and  $\gamma_r : \mathfrak{s}_r \otimes \mathfrak{s}_r \rightarrow V$ . There are two possible choices of  $S$  :

$$S^* = \mathfrak{s}_l \otimes W, \Gamma = \gamma_l \otimes \omega; \quad S^* = \mathfrak{s}_r \otimes W, \Gamma = \gamma_r \otimes \omega, \dim S = 8 \times 16^n.$$

Automorphism Lie algebra  $\text{aut} = \mathfrak{so}(8n + 6) \times \mathfrak{sl}(2)$ .

- $\dim V = 8n + 7$

The intertwiner  $\gamma : \mathfrak{s} \otimes \mathfrak{s} \rightarrow V$  is antisymmetric.

$$S = S^* = \mathfrak{s} \otimes W, \Gamma = \gamma \otimes \omega, \dim S = 16 \times 16^n.$$

Automorphism Lie algebra  $\text{aut} = \mathfrak{so}(8n + 7) \times \mathfrak{sl}(2)$ .

One can consider also  $N$ -extended supersymmetry Lie algebra. This means that we should start with reducible spinorial representation  $S_N$  (direct sum of  $N$  copies of spinorial representation  $S$ ). Taking  $N$  copies of the intertwiner  $V \rightarrow \text{Sym}^2 S$  we obtain an intertwiner  $V \rightarrow \text{Sym}^2 S_N$ . We define the  $N$ -extended supersymmetry Lie algebra by means of this intertwiner. The Lie algebra acting on its cohomology acquires an additional factor  $\mathfrak{gl}(N)$ .

Notice that in the cases when there are two different possible choices of  $S$  (denoted by  $S_1$  and  $S_2$ ) one can talk about  $(N_1, N_2)$ -extended supersymmetry taking as a starting point a direct sum of  $N_1$  copies of  $S_1$  and  $N_2$  copies of  $S_2$ .

The description of cohomology of supersymmetry Lie algebras in dimensions 9, 8, 7 follows immediately from the description of cohomology of ten-dimensional supersymmetry Lie algebra reduced to these dimensions. (Notice  $S$  has dimension 16 in all of these cases.)

We will describe the cohomology of the Lie algebra of supersymmetries in six-dimensional case as representations of the Lie algebra  $\mathfrak{so}(6) \times \mathfrak{sl}(2)$ . The vector representation  $V$  of  $\mathfrak{so}(6)$  has the highest weight  $[1, 0, 0]$ , the irreducible spinor representations have highest weights  $[0, 0, 1]$ ,  $[0, 1, 0]$ ; we consider for definiteness  $\mathfrak{sl}$  with highest weight  $[0, 0, 1]$ . As a representation  $\mathfrak{so}(6) \times \mathfrak{sl}(2)$  the representation  $V$  has the weight  $[1, 0, 0, 0]$  and the representation  $S = \mathfrak{sl} \otimes W$  has the weight  $[0, 0, 1, 1]$ . The description of graded component of cohomology group with gradings  $k = m + 2n$  and  $n$  is given by the formulas

$$H^{k,0} = [0, 0, k, k] \quad (81)$$

$$H^{k,1} = [1, 0, k-3, k-2] \quad (82)$$

$$H^{k,2} = [0, 1, k-6, k-4] \quad (83)$$

$$H^{k,3} = [0, 0, k-8, k-6] \quad (84)$$

The only special case is when  $k = 4$ , there is one additional term, a scalar, for  $H^{4,1}$ .

$$H^{4,1} = [0, 0, 0, 0] \oplus [1, 0, 1, 2] \quad (85)$$

For  $n \geq 4$ ,  $H^{k,n}$  vanishes. The  $\mathfrak{so}(6) \times \mathfrak{sl}(2)$ -invariant part is in  $H^{0,0}$ , and  $H^{4,1}$ .

The dimensions of the cohomology groups are encoded in Poincare series:

$$P_0(\tau) = \frac{1+3\tau}{(1-\tau)^5}, \quad (86)$$

$$P_1(\tau) = \frac{8\tau^3 + 6\tau^4 - 6\tau^5 + 10\tau^6 - 10\tau^7 + 5\tau^8 - \tau^9}{(1-\tau)^5}, \quad (87)$$

$$P_2(\tau) = \frac{18\tau^6 - 10\tau^7 + 5\tau^8 - \tau^9}{(1-\tau)^5}, \quad (88)$$

$$P_3(\tau) = \frac{3\tau^8 + \tau^9}{(1-\tau)^5} \quad (89)$$

The cohomology considered as  $\mathbb{C}[t^1, \dots, t^\alpha, \dots]$ -module is generated by the scalar and

$$\begin{aligned} [t^\alpha c_m \Gamma_{\alpha\beta}^m] &\in H^{3,1}, \\ [t^\alpha t^\beta c_m c_n \Gamma_{\alpha\beta}^{mnk}] &\in H^{6,2}, \\ [t^\alpha t^\beta c_m c_n c_k \Gamma_{\alpha\beta}^{mnk}] &\in H^{8,3} \end{aligned}$$

Now we will describe the cohomology of the Lie algebra of supersymmetries in five-dimensional case as representations of the Lie algebra  $\mathfrak{so}(5) \times \mathfrak{sl}(2)$ . The vector representation  $V$  of  $\mathfrak{so}(5)$  has the highest weight  $[1, 0]$ , the irreducible spinorial representation has highest weight  $[0, 1]$ . As a representation  $\mathfrak{so}(5) \times \mathfrak{sl}(2)$  the representation  $V$  has the weight  $[1, 0, 0]$  and the representation  $S = \mathfrak{s} \otimes W$  has the weight  $[0, 1, 1]$ . The description of graded component of cohomology group with gradings  $k = m + 2n$  and  $n$  is given by the following formulas (for  $n \geq 3$ ,  $H^{k,n}$  vanishes)

$$H^{k,0} = [0, k, k] \quad (90)$$

$$H^{k,1} = [1, k-4, k-2] \quad (91)$$

$$H^{k,2} = [0, k-6, k-4] \quad (92)$$

The only special case is when  $k = 2$ , there is one additional term, a scalar, for  $H^{2,1}$ .

$$H^{2,1} = [0, 0, 0] \oplus [0, 2, 2] \quad (93)$$

The  $\mathfrak{so}(5) \times \mathfrak{sl}(2)$ -invariant part is in  $H^{0,0}$ , and  $H^{2,1}$ .

The dimensions of the cohomology groups are encoded in Poincare series:

$$P_0(\tau) = \frac{1 + 3\tau + \tau^2 - 5\tau^3 + 10\tau^4 - 10\tau^5 + 5\tau^6 - \tau^7}{(1 - \tau)^5}, \quad (94)$$

$$P_1(\tau) = \frac{15\tau^4 - 11\tau^5 + 5\tau^6 - \tau^7}{(1 - \tau)^5}, \quad (95)$$

$$P_2(\tau) = \frac{3\tau^6 + \tau^7}{(1 - \tau)^5} \quad (96)$$

The cohomology regarded as  $\mathbb{C}[t^1, \dots, t^\alpha, \dots]$ -module is generated by the scalar considered as an element of  $H^{0,0}$  and

$$\begin{aligned} [t^\alpha t^\beta c_m \Gamma_{\alpha\beta}^{mn}] &\in H^{4,1}, \\ [t^\alpha t^\beta c_m c_n \Gamma_{\alpha\beta}^{mn}] &\in H^{6,2} \end{aligned}$$

In four-dimensional case the representation  $S$  should be considered as 4-dimensional Dirac spinor.

We describe the cohomology of the Lie algebra of supersymmetries in four-dimensional case as representations of the Lie algebra  $\mathfrak{so}(4)$ . As usual the representations are labeled by their highest weight. The vector representation  $V$  has the highest weight  $[1, 1]$ , the irreducible spinor representations have highest weights  $\mathfrak{s}_l = [0, 1]$ ,  $\mathfrak{s}_r = [1, 0]$ ; we assume that  $S = \mathfrak{s}_l + \mathfrak{s}_r = [0, 1] \oplus [1, 0]$ . The description of graded component of cohomology group with gradings  $k = m + 2n$  and  $n$  is given by the following formulas (for  $n \geq 6$ ,  $H^{k,n}$  vanishes)

$$H^{k,0} = [0, k] \oplus [k, 0] \quad (97)$$

$$H^{k,1} = [1, k-3] \oplus [k-3, 1] \quad (98)$$

$$H^{k,2} = [0, k-6] \oplus [k-6, 0], \quad (99)$$

The only special case is when  $k = 4$ , there is one additional term, a scalar, for  $H^{4,1}$ .

$$H^{4,1} = [0, 0] \oplus 2 \times [1, 1] \quad (100)$$

The  $\mathfrak{so}(4)$ -invariant part is in  $H^{0,0}$ ,  $H^{6,2}$ , and  $H^{4,1}$ .



The dimensions of these cohomology groups are encoded in Poincare series:

$$P_0(\tau) = \frac{1 + 2\tau - \tau^2}{(1 - \tau)^2}, \quad (101)$$

$$P_1(\tau) = \frac{4\tau^3 + \tau^4 - 2\tau^5 + \tau^6}{(1 - \tau)^2}, \quad (102)$$

$$P_2(\tau) = \frac{2\tau^6}{(1 - \tau)^2} \quad (103)$$

The cohomology can be regarded as  $\mathbb{C}[t^\alpha, t^{\dot{\alpha}}]$ -module where  $\alpha = 1, 2$ ,  $\dot{\alpha} = \dot{1}, \dot{2}$ . (Here  $t^\alpha$  transforms according to the representation  $[1, 0]$  and  $t^{\dot{\alpha}}$  transforms according to the representation  $[0, 1]$ .) This module is generated by the scalar considered as an element of  $H^{0,0}$  and

$$\begin{aligned} [t^\alpha c_m \Gamma_{\alpha\dot{\beta}}^m] &\in H^{3,1}, \\ [t^{\dot{\alpha}} c_m \Gamma_{\dot{\alpha}\beta}^m] &\in H^{3,1}, \\ [t^\alpha t^{\dot{\beta}} c_m c_n \Gamma_{\alpha\dot{\beta}}^{mn}] &\in H^{6,2}, \\ [t^{\dot{\alpha}} t^\beta c_m c_n \Gamma_{\dot{\alpha}\beta}^{mn}] &\in H^{6,2} \end{aligned}$$

The cohomology generators in  $D = 4$  and  $D = 5$  were found by F. Brandt [6].

## 7 Calculations

We do calculation of Poincare series applying [10]. However, the straightforward calculation is pretty lengthy with the computers we are using. Therefore for  $D = 10$  and  $D = 11$  we consider dimensional reduction to dimension  $r$  and we are using induction with respect to  $r$ .

Recall that the differential of  $D$ -dimensional theory reduced to dimension  $r$  has the form

$$d_r = \sum_{1 \leq m \leq r} A^m \frac{\partial}{\partial c^m}. \quad (104)$$

where  $A^m = \frac{1}{2} \Gamma_{\alpha\beta}^m t^\alpha t^\beta$  and acts in the space  $E_r$  of polynomial functions of even ghosts  $t^\alpha$  and  $r$  odd ghosts  $c^1, \dots, c^r$ . We denote corresponding cohomology by  $H_r$ . Both  $E_r$  and  $H_r$  are bigraded by the degree of even ghosts  $m$  and

degree of odd ghosts  $n$ , but it is simpler to work with grading with respect to  $k = m + 2n$  and  $n$ . An element of  $E_r$  can be represented in the form  $x + yc^r$  where  $x, y \in E_{r-1}$ . Notice that

$$d_r(x + yc^r) = d_{r-1}x + A^r y + d_{r-1}yc^r.$$

The operator of multiplication by  $A_m$  commutes with the differential, hence it induces a homomorphism

$$\sigma : H_{r-1} \rightarrow H_{r-1}.$$

Sending  $x \in E_{r-1}$  into  $x + 0c^m \in E_r$  (embedding  $E_{r-1}$  into  $E_r$ ) we obtain a homomorphism  $H_{r-1} \rightarrow H_r$ . Sending  $x + yc^m$  into  $y$  we get a homomorphism  $H_r \rightarrow H_{r-1}$ . It is easy to see that combining these homomorphisms we obtain an exact sequence

$$H_{r-1} \rightarrow H_{r-1} \rightarrow H_r \rightarrow H_{r-1} \rightarrow H_{r-1}$$

or, taking into account the gradings,

$$\dots \rightarrow H_{r-1}^{k,n} \rightarrow H_{r-1}^{k+2,n} \rightarrow H_r^{k+2,n} \rightarrow H_{r-1}^{k,n-1} \rightarrow H_{r-1}^{k+2,n-1} \rightarrow \dots \quad (105)$$

(This is the exact sequence of a pair  $(E_r, E_{r-1})$ ; we use the fact that  $E_r^{k,n}/E_{r-1}^{k,n} \cong E_{r-1}^{k-2,n-1}$ .) It follows immediately from this exact sequence that an isomorphism  $H_{r-1}^{k,n} = 0$  for  $n > n_{r-1}$  implies  $H_r^{k,n} = 0$  for  $n > n_r + 1$ . (In other words if  $n_r$  is the maximal degree of cohomology in  $r$ -dimensional reduction then  $n_{r+1} \leq n_r + 1$ .) Applying the exact sequence (105) to the case  $n = n_r$  and assuming that  $n_{r-1} < n_r$  we obtain an isomorphism between  $H_r^{k+2,n_r}$  and a subgroup of  $H_{r-1}^{k,n_{r-1}}$  (this isomorphism can be considered as an isomorphism of  $\mathfrak{so}(r-1)$ -representations). For dimensional reductions of  $D = 10$  and  $D = 11$  algebras of supersymmetries dimensions of  $H_r^{k+2,n_r}$  and  $H_{r-1}^{k,n_{r-1}}$  coincide because Poincare series are related by the formula  $P_{n_r} = \tau^2 P_{n_{r-1}}$ . If homomorphism  $\sigma$  is injective we obtain a short exact sequence

$$0 \rightarrow H_{r-1}^{k,n} \rightarrow H_{r-1}^{k+2,n} \rightarrow H_r^{k+2,n} \rightarrow 0.$$

Calculations with [10] show that  $n_1 = \dots = n_5 = 0$  for  $D = 10$  and  $n_1 = \dots = n_9 = 0$  for  $D = 11$ . (It is sufficient to check that in corresponding dimensions the homomorphism  $\sigma$  is injective.)

To analyze  $r$ -dimensional reduction for  $r > 5, D = 10$  we notice that  $d_r$  can be considered as a sum of differentials  $d'$  and  $d''$  where

$$d' = \sum_{1 \leq m \leq 5} A^m \frac{\partial}{\partial c^m},$$

$$d'' = \sum_{5 < m \leq r} A^m \frac{\partial}{\partial c^m}.$$

(For  $r > 9, D = 11$  one should replace 5 by 9.) These differentials anticommute; this allows us to use the spectral sequence of bicomplex to calculate the cohomology of  $d_r$ . The spectral sequence of bicomplex starts with cohomology  $H(d'', H(d'))$ . Taking into account that the cohomology  $H(d') = H(d_5)$  is concentrated in degree 0 (as the cohomology  $H_5$ ) we obtain that the spectral sequence terminates. This means that one can calculate the Poincare series of  $d_r$  as the Poincare series of  $H(d'', H(d'))$  using [10]. Again applying [10] we can obtain the information about generators of cohomology; this information is sufficient to express the generators in terms of Gamma-matrices.

To calculate the cohomology as a representation of the group of automorphisms we decompose each graded component  $E^{k,n} = \text{Sym}^{k-2n} S \otimes \Lambda^n V$  of  $E$  into direct sum of irreducible representations.

For example, for  $D = 10$  spacetime, we have the cochain complex

$$\begin{aligned} 0 &\xleftarrow{d_0} \text{Sym}^k S \xleftarrow{d_1} \text{Sym}^{k-2} S \otimes V \xleftarrow{d_2} \text{Sym}^{k-4} S \otimes \wedge^2 V \\ &\xleftarrow{d_3} \text{Sym}^{k-6} S \otimes \wedge^3 V \xleftarrow{d_4} \text{Sym}^{k-8} S \otimes \wedge^4 V \xleftarrow{d_5} \text{Sym}^{k-10} S \otimes \wedge^5 V \\ &\xleftarrow{d_6} \text{Sym}^{k-12} S \otimes \wedge^6 V \xleftarrow{d_7} \text{Sym}^{k-14} S \otimes \wedge^7 V \xleftarrow{d_8} \text{Sym}^{k-16} S \otimes \wedge^8 V \\ &\xleftarrow{d_9} \text{Sym}^{k-18} S \otimes \wedge^9 V \xleftarrow{d_{10}} \text{Sym}^{k-20} S \otimes \wedge^{10} V \xleftarrow{d_{11}} 0 \end{aligned} \quad (106)$$

where for  $\text{Sym}^m S \otimes \wedge^n V$ , the grading index  $k = m + 2n$  is preserved by  $d$ . All components of this complex can be regarded as representations of  $\mathfrak{so}(10)$ . We

have

$$\begin{aligned}
S &= [0, 0, 0, 0, 1] \text{ (choosen) or } [0, 0, 0, 1, 0], \quad V = [1, 0, 0, 0, 0] \\
\wedge^2 V &= [0, 1, 0, 0, 0], \quad \wedge^3 V = [0, 0, 1, 0, 0], \\
\wedge^4 V &= [0, 0, 0, 1, 1], \quad \wedge^5 V = [0, 0, 0, 0, 2] \oplus [0, 0, 0, 2, 0], \\
\wedge^6 V &= \wedge^4 V, \quad \wedge^7 V = \wedge^3 V, \quad \wedge^8 V = \wedge^2 V, \quad \wedge^9 V = V, \quad \wedge^{10} V = [0, 0, 0, 0, 0],
\end{aligned} \tag{107}$$

$$\text{Sym}^k S = \bigoplus_{i=0}^{[k/2]} [i, 0, 0, 0, k-2i] \tag{108}$$

(see [15] for the decomposition of  $\text{Sym}^m S \otimes \wedge^n V$  and for complete description of action of differential on irreducible components for supersymmetry Lie algebra in 10D and 6D.)

By the Schur's lemma an intertwiner between irreducible representations (a homomorphism of simple modules) is either zero or an isomorphism. This means that an intertwiner between non-equivalent irreducible representations always vanishes. This observation permits us to calculate the contribution of every irreducible representation to the cohomology separately.

Let us fix an irreducible representation  $A$  and the number  $k$ . We will denote by  $\nu_n$  (or by  $\nu_n^k$  if it is necessary to show the dependence of  $k$ ) the multiplicity of  $A$  in  $E^{k,n} = \text{Sym}^{k-2n} S \otimes \wedge^n V$ . The multiplicity of  $A$  in the image of  $d : E^{k,n} \rightarrow E^{k,n-1}$  will be denoted by  $\kappa_n$ , then the multiplicity of  $A$  in the kernel of this map is equal to  $\nu_n - \kappa_n$  and the multiplicity of  $A$  in the cohomology  $H^{k,n}$  is equal to  $h_n = \nu_n - \kappa_n - \kappa_{n+1}$ . It follows immediately that the multiplicity of  $A$  in virtual representation  $\sum_n (-1)^n H^{k,n}$  (in the Euler characteristic) is equal to  $\sum (-1)^n \nu_n$ . It does not depend on  $\kappa_n$ , however, to calculate the cohomology completely we should know  $\kappa_n$ .

Let us consider as an example  $A = [0, 0, 0, 0, 0]$ , the scalar representation, for dimension  $D = 10$  and arbitrary  $k$ . For all  $k \neq 4, 12$ , we have  $\nu_i = 0$ . (For small  $k$  this can be obtained by means of LiE program [9].) For  $k = 4$ , we have all  $\nu_i$  vanish except  $\nu_1 = 1$ , hence all  $\kappa_i$  vanish. The multiplicity of  $[0, 0, 0, 0, 0]$  in  $H^{4,1}$  is equal to 1, and other cohomology  $H^{4,i}$  do not contain

scalar representation. For  $k = 12$ , all  $\nu_i$  vanish except  $\nu_5 = 1$ , hence  $H^{12,5}$  contains  $[0, 0, 0, 0, 0]$  with multiplicity 1, and  $H^{12,i}$  do not contain  $[0, 0, 0, 0, 0]$  for  $i \neq 5$ . This agrees with Eq. 9 and Eq. 8, respectively.

In many cases a heuristic calculation of cohomology can be based on a principle that kernel should be as small as possible; in other words, the image should be as large as possible (this is an analog of the general rule of the physics of elementary particles: Everything happens unless it is forbidden). In [8] this is called the principle of maximal propagation.<sup>5</sup> Let us illustrate this principle in the case when  $k = 9$  and  $A = [0, 1, 0, 0, 1]$  in  $10D$ . In this case  $\nu_4 = 1$ ,  $\nu_3 = 3$ ,  $\nu_2 = 1$ . If we believe in the maximal propagation, then  $\kappa_3 = 1$ ,  $\kappa_4 = 1$ , thus we have  $\nu_3 - \kappa_3 - \kappa_4 = 1$ , and  $[0, 1, 0, 0, 1]$  contributes only to  $H^{9,3}$ .

Notice, that the principle of maximal propagation sometimes does not give a definite answer. For example, in the case when  $k = 8$  in the dimension reduced to 7 from  $10D$ . Considering only the multiplicities of  $A = [0, 0, 2, 2]$ , we have sequence  $0 \rightarrow 0 \rightarrow 2 \rightarrow 5 \rightarrow 3 \rightarrow 1$ . This sequence offers two distinct possibilities even under the assumption of maximal propagation. We can assume that the kernels of the differentials  $2 \rightarrow 5$  and  $5 \rightarrow 3$  are minimal. In this case  $h_0 = 1$ . Or we can start with assumption that the kernel of differential  $3 \rightarrow 1$  is minimal, then the kernel of  $5 \rightarrow 3$  has multiplicity at least 3 and assuming that this multiplicity is equal to 3 we see under the assumption of minimality of the kernel of  $2 \rightarrow 5$  that the only non-trivial cohomology is  $h_2 = 1$ . (We can prove that the second position is the correct choice.) We see that the result of the method of maximal propagation depends on what differential we choose to start with. More generally, we should order the differentials in the complex in some way and apply the principle of maximal propagation using this ordering.

In cases we are interested in one can prove that the principle of maximal propagation augmented with information about Poincare series and generators is free of ambiguities. In other words, this information permits us to resolve

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<sup>5</sup> Notice that the principle of maximal propagation should be applied to the decomposition of cohomology into irreducible representations of the full automorphism group. Otherwise we do not use all available information.

ambiguities in the application of this principle. Sometimes it is useful to apply the remark that multiplying a coboundary by a polynomial we again obtain a coboundary.

The only exception is the case of ten-dimensional reduction of eleven-dimensional supersymmetry Lie algebra. In this case we use the isomorphism between  $H_{10}^{k,1}$  and  $H_{11}^{k-2,2}$  that was derived from Eq. 105. This is an isomorphism of  $\mathfrak{so}(10)$ -representations; it allows us to find the decomposition of  $H_{10}^{k,1}$  from decomposition of  $H_{11}^{k-2,2}$  in irreducible representations of  $\mathfrak{so}(11)$ . From the other side we can find the virtual  $\mathfrak{so}(10)$ -character of  $H_{10}^{k,0} - H_{10}^{k,1}$  (Euler characteristic); this allows us to finish the calculation.

Let  $\cdots \rightarrow M_n \rightarrow \cdots \rightarrow M_0 \rightarrow M \rightarrow 0$  denote the minimal free  $\sum \text{Sym}^m S$ -resolution of the module  $M = \sum_k H^{k,n}$ . Then every free module  $M_i$  in this resolution is a representation of the group of automorphisms  $Aut$ ; it is a tensor product of a finite dimensional graded representation  $\mu_i$  (the generators) and a representation of  $Aut$  in polynomial algebra  $\sum \text{Sym}^m S$ . It is easy to find the dimensions of  $Aut$ -modules  $\mu_i$  (the number of generators of  $M_i$ ) using [10]. Dimensions of the graded components of  $\mu_i$  can be found routinely using [10].

The information about free resolution can be used to find the structure of  $Aut$ -module on  $\mu_i$  and therefore on  $M$ . However, we went in opposite direction: we used the information about the structure of  $Aut$ -module on  $M$  to find the structure of  $Aut$ -module on  $\mu_i$  using the formula <sup>6</sup>

$$\sum_i (-1)^i \mu_i = \left( \sum_k H^{k,n} \tau^k \right) \otimes \left( \sum (-\tau)^j \Lambda^j S \right). \quad (109)$$

The analysis of the resolution of the cohomology module is relegated to the appendix.

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<sup>6</sup>This formula follows from well known  $K$ -theory relation

$$\left( \sum \tau^m \text{Sym}^m S \right) \left( \sum (-\tau)^j \Lambda^j S \right) = 1.$$

Taking into account that parity reversal transforms symmetric power into exterior power we can understand this relation in the framework of super algebra.

## 8 Homology of super Poincare Lie algebra

The super Poincare Lie algebra can be defined as super Lie algebra spanned by supersymmetry Lie algebra and Lie algebra *aut* of its group of automorphisms *Aut*.<sup>7</sup>

To calculate the homology and cohomology of super Poincare Lie algebra we will use the following statement proved by Hochschild and Serre [12] .(It follows from Hochschild-Serre spectral sequence constructed in the same paper.)

Let  $\mathcal{P}$  denote a Lie algebra represented as a vector space as a direct sum of two subspaces  $\mathcal{L}$  and  $\mathcal{G}$ . We assume that  $\mathcal{G}$  is an ideal in  $\mathcal{P}$  and that  $\mathcal{L}$  is semisimple. It follows from the assumption that  $\mathcal{G}$  is an ideal that  $\mathcal{L}$  acts on  $\mathcal{G}$  and therefore on cohomology of  $\mathcal{G}$ ; the  $\mathcal{L}$ -invariant part of cohomology  $H^\bullet(\mathcal{G})$  will be denoted by  $H^\bullet(\mathcal{G})^\mathcal{L}$ . One can prove that

$$H^n(\mathcal{P}) = \sum_{p+q=n} H^p(\mathcal{L}) \otimes H^q(\mathcal{G})^\mathcal{L}.$$

This statement remains correct if  $\mathcal{P}$  is a super Lie algebra. We will apply it to the case when  $\mathcal{P}$  is super Poincare Lie algebra,  $\mathcal{G}$  is the Lie algebra of supersymmetries and  $\mathcal{L}$  is the Lie algebra of automorphisms or its semisimple subalgebra. (We are working with complex Lie algebras, but we can work with their real forms. The results do not change.)

Notice that it is easy to calculate the cohomology of semisimple Lie algebra  $\mathcal{L}$ ; they are described by antisymmetric tensors on  $\mathcal{L}$  that are invariant with respect to adjoint representation. One can say also that they coincide with de Rham cohomology of corresponding compact Lie group. The Lie algebra cohomology of  $L = \mathfrak{so}_{10}$  with trivial coefficients and as well as De Rham cohomology of the compact Lie group  $SO(10, \mathbb{R})$  is a Grassmann algebra with generators of dimension 3,7,11,13 and 9. In general the cohomology of the group  $SO(2r, \mathbb{R})$  is a Grassmann algebra with generators  $e_i$  having dimension  $4i - 1$  for  $i < r$  and the dimension  $2r - 1$  for  $i = r$ . The cohomology of the group  $SO(2r + 1, \mathbb{R})$

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<sup>7</sup>Instead of Lie algebra of automorphisms one can take its subalgebra. For example, we can take as a subalgebra the orthogonal Lie algebra

is a Grassmann algebra with generators  $e_i$  having dimension  $4i - 1$  for  $i \leq r$ . The cohomology of Lie algebra  $\mathfrak{sl}(n)$  coincide with the cohomology of compact Lie group  $SU(n)$ ; they form a Grassmann algebra with generators of dimension  $3, 5, \dots, 2n - 1$ .

As we have seen only  $\mathcal{L}$ -invariant part of cohomology of Lie algebra of supersymmetries contributes to the cohomology of super Poincare algebra. For  $D = 10$  this means that the only contribution comes from subspaces  $\text{Sym}^m S \otimes \Lambda^n V$  having the following indices  $(m, n) = (0, 0)$ ,  $(m, n) = (2, 1)$  and  $(m, n) = (2, 5)$ , for  $D = 11$  the only contribution comes from  $(m, n) = (0, 0)$  and  $(m, n) = (2, 2)$ , for  $D = 6$  the only contribution comes from  $(m, n) = (0, 0)$  and  $(m, n) = (2, 1)$ . (Here  $m = k - 2n$  denotes the grading with respect to even ghosts  $t^\alpha$  and  $n$  the grading with respect to odd ghosts  $c_m$ .)

Cocycles representing cohomology classes of super Poincare algebra can be written in the form  $\rho \otimes h$ , where  $\rho$  is an invariant antisymmetric tensor with respect to adjoint representation of  $aut$  and  $h$  is 1 or

$$t^\alpha t^\beta c_m c_n \Gamma_{\alpha\beta}^{mn} \text{ for } D = 11 \quad (110)$$

$$t^\alpha t^\beta c_m \Gamma_{\alpha\beta}^m, \quad t^\alpha t^\beta c_m c_n c_k c_l c_r \Gamma_{\alpha\beta}^{mnklr} \text{ for } D = 10 \quad (111)$$

$$t^\alpha t^\beta, \quad t^\alpha t^\beta c_m c_n c_k c_l \Gamma_{\alpha\beta}^{mnkl} \text{ for } D = 9 \quad (112)$$

$$t^\alpha t^\beta c_m c_n c_k \Gamma_{\alpha\beta}^{mnk} \text{ for } D = 8 \quad (113)$$

$$t^\alpha t^\beta c_m c_n \Gamma_{\alpha\beta}^{mn} \text{ for } D = 7 \quad (114)$$

$$t^\alpha t^\beta c_m \Gamma_{\alpha\beta}^m, \quad t^\alpha t^\beta c_m c_n c_k \Gamma_{\alpha\beta}^{mnk} \text{ for } D = 6 \quad (115)$$

$$t^\alpha t^\beta c_m c_n \Gamma_{\alpha\beta}^{mn} \text{ for } D = 5 \quad (116)$$

$$t^\alpha t^{\dot{\beta}} c_m \Gamma_{\alpha\dot{\beta}}^m, t^{\dot{\alpha}} t^\beta c_m \Gamma_{\dot{\alpha}\beta}^m, t^\alpha t^{\dot{\beta}} c_m c_n \Gamma_{\alpha\dot{\beta}}^{mn}, t^{\dot{\alpha}} t^\beta c_m c_n \Gamma_{\dot{\alpha}\beta}^{mn} \text{ for } D = 4. \quad (117)$$

Here Greek indices (i.e. spinor indices) take values  $1, 2, \dots, \dim S$  and Roman indices (i.e. vector indices) take values  $1, 2, \dots, D$ , and  $\dim S$  is defined in Section.6. The only exception is for  $D = 4$ , the Greek indices  $\alpha, \beta$  take values  $1, 2$ , and the dotted Greek indices  $\dot{\alpha}, \dot{\beta}$  take values  $\dot{1}, \dot{2}$ . Notice that in these formulas Gamma matrices and summation range depend on the choice of dimension.



The general definition of super Poincare algebra can be applied also to reduced supersymmetry Lie algebra. For  $D = 10$  and  $D = 11$  the role of super Poincare Lie algebra is played by the semidirect product of reduced supersymmetry Lie algebra and  $\mathfrak{so}(r) \times \mathfrak{so}(D - r)$ . The information about invariant elements provided in Sections 4 and 5 permits us to describe cohomology of this generalization of super Poincare algebra.

## A Resolution of the cohomology modules

One can find a minimal free resolution of the  $R$ -module  $\sum_k H^{k,n} = M$ . (Here  $R = \mathbb{C}[t^1, \dots, t^\alpha, \dots] = \sum_m \text{Sym}^m S$ .) The reader may wish to consult [1] on this subject. The free resolution has the form

$$\cdots \rightarrow M_i \rightarrow \cdots \rightarrow M_0 \rightarrow M \rightarrow 0$$

where  $M_i = \mu_i \otimes R$ , and

$\mu_0$  - generators of  $M$ ;

$\mu_1$  - relations between generators of  $M$ ;

$\mu_2$  - relations between relations ;

$\cdots$

### A.1 Resolution of the cohomology modules of dimensional reduction of 10D Lie algebra of supersymmetries

We give the structure of  $\mu_i$  as *aut*-module and its grading in the case of  $r$ -dimensional reduction of ten-dimensional Lie algebra of supersymmetries (in the case when  $D = r + (10 - r)$ ). Recall that *aut* denotes the Lie algebra of the group of automorphisms *Aut*.

- D=10+0, n=0

$$\mu_0 = [0, 0, 0, 0, 0], \dim(\mu_0) = 1, \deg(\mu_0) = 0;$$

$$\mu_1 = [1, 0, 0, 0, 0], \dim(\mu_1) = 10, \deg(\mu_1) = 2;$$

$$\mu_2 = [0, 0, 0, 1, 0], \dim(\mu_2) = 16, \deg(\mu_2) = 3;$$

$$\mu_3 = [0, 0, 0, 0, 1], \dim(\mu_3) = 16, \deg(\mu_3) = 5;$$

$$\mu_4 = [1, 0, 0, 0, 0], \dim(\mu_4) = 10, \deg(\mu_4) = 6;$$

$$\mu_5 = [0, 0, 0, 0, 0], \dim(\mu_5) = 1, \deg(\mu_5) = 8.$$

- D=10+0, n=1

$$\mu_0 = [0, 0, 0, 1, 0], \dim(\mu_0) = 16, \deg(\mu_0) = 3;$$

$$\mu_1 = \mu_1' + \mu_1'',$$

$$\mu_1' = [0, 1, 0, 0, 0], \dim(\mu_1') = 45, \deg(\mu_1') = 4;$$

$$\mu_1'' = [0, 0, 0, 0, 1], \dim(\mu_1'') = 16, \deg(\mu_1'') = 5;$$

$$\mu_2 = 2 \times [0, 0, 1, 0, 0] + [1, 0, 0, 0, 0], \dim(\mu_2) = 250, \deg(\mu_2) = 6;$$

$$\mu_3 = [0, 0, 0, 1, 0] + [0, 1, 0, 1, 0] + [1, 0, 0, 0, 1], \dim(\mu_3) = 720, \deg(\mu_3) = 7;$$

$$\mu_4 = \mu_4' + \mu_4'',$$

$$\mu_4' = [0, 2, 0, 0, 0] + [1, 0, 0, 2, 0] + [2, 0, 0, 0, 0], \dim(\mu_4') = 1874, \deg(\mu_4') = 8;$$

$$\mu_4'' = [0, 0, 0, 0, 1], \dim(\mu_4'') = 16, \deg(\mu_4'') = 9;$$

$$\mu_5 = \mu_5' + \mu_5'',$$

$$\mu_5' = [0, 0, 0, 0, 1] + [0, 0, 0, 0, 1] + [0, 0, 0, 3, 0] + [1, 1, 0, 1, 0], \dim(\mu_5') = 4352, \deg(\mu_5') = 9;$$

$$\mu_5'' = [1, 0, 0, 0, 0], \dim(\mu_5'') = 9, \deg(\mu_5'') = 10;$$

$$\mu_6 = [0, 1, 0, 2, 0] + [2, 0, 1, 0, 0], \dim(\mu_6) = 8008, \deg(\mu_6) = 10;$$

$$\mu_7 = [1, 0, 1, 1, 0] + [3, 0, 0, 0, 1], \dim(\mu_7) = 11440, \deg(\mu_7) = 11;$$

$$\mu_8 = [0, 0, 2, 0, 0] + [2, 0, 0, 1, 1] + [4, 0, 0, 0, 0], \dim(\mu_8) = 12870, \deg(\mu_8) = 12;$$

$$\mu_9 = [1, 0, 1, 0, 1] + [3, 0, 0, 1, 0], \dim(\mu_9) = 11440, \deg(\mu_9) = 13;$$

$$\mu_{10} = [0, 1, 0, 0, 2] + [2, 0, 1, 0, 0], \dim(\mu_{10}) = 8008, \deg(\mu_{10}) = 14;$$

$$\mu_{11} = [0, 0, 0, 0, 3] + [1, 1, 0, 0, 1], \dim(\mu_{11}) = 4368, \deg(\mu_{11}) = 15;$$

$$\mu_{12} = [0, 2, 0, 0, 0] + [1, 0, 0, 0, 2], \dim(\mu_{12}) = 1820, \deg(\mu_{12}) = 16;$$

$$\mu_{13} = [0, 1, 0, 0, 1], \dim(\mu_{13}) = 560, \deg(\mu_{13}) = 17;$$

$$\mu_{14} = [0, 0, 1, 0, 0], \dim(\mu_{14}) = 120, \deg(\mu_{14}) = 18;$$

$$\mu_{15} = [0, 0, 0, 1, 0], \dim(\mu_{15}) = 16, \deg(\mu_{15}) = 19;$$

$$\mu_{16} = [0, 0, 0, 0, 0], \dim(\mu_{16}) = 1, \deg(\mu_{16}) = 20.$$

- D=10+0, n=2

$$\mu_0 = [0, 0, 1, 0, 0], \dim(\mu_0) = 120, \deg(\mu_0) = 6;$$

$$\mu_1 = [0, 0, 0, 1, 0] + [0, 1, 0, 1, 0] + [1, 0, 0, 0, 1], \dim(\mu_1) = 720, \deg(\mu_1) = 7;$$

$$\mu_2 = [0, 0, 0, 0, 0] + [0, 0, 0, 1, 1] + [0, 1, 0, 0, 0] + [0, 2, 0, 0, 0] + [1, 0, 0, 2, 0] + [2, 0, 0, 0, 0], \dim(\mu_2) = 2130, \deg(\mu_2) = 8;$$

$$\mu_3 = \mu_3' + \mu_3'',$$

$$\mu_3' = [0, 0, 0, 3, 0] + [1, 0, 0, 1, 0] + [1, 1, 0, 1, 0], \dim(\mu_3') = 4512, \deg(\mu_3') = 9;$$

$$\mu_3'' = [0, 0, 0, 0, 2] + [1, 0, 0, 0, 0], \dim(\mu_3'') = 136, \deg(\mu_3'') = 10;$$

$$\mu_4 = \mu_4' + \mu_4'',$$

$$\mu_4' = [0, 1, 0, 2, 0] + [2, 0, 1, 0, 0], \dim(\mu_4') = 8008, \deg(\mu_4') = 10;$$

$$\mu_4'' = [0, 0, 0, 1, 0] + [1, 0, 0, 0, 1], \dim(\mu_4'') = 160, \deg(\mu_4'') = 11;$$

$$\mu_5 = \mu_5' + \mu_5'',$$

$$\mu_5' = [1, 0, 1, 1, 0] + [3, 0, 0, 0, 1], \dim(\mu_5') = 11440, \deg(\mu_5') = 11;$$

$$\mu_5'' = [0, 1, 0, 0, 0], \dim(\mu_5'') = 45, \deg(\mu_5'') = 12;$$

$$\mu_6 = [0, 0, 2, 0, 0] + [2, 0, 0, 1, 1] + [4, 0, 0, 0, 0], \dim(\mu_6) = 12870, \deg(\mu_6) = 12;$$

$$\mu_7 = [1, 0, 1, 0, 1] + [3, 0, 0, 1, 0], \dim(\mu_7) = 11440, \deg(\mu_7) = 13;$$

$$\mu_8 = [0, 1, 0, 0, 2] + [2, 0, 1, 0, 0], \dim(\mu_8) = 8008, \deg(\mu_8) = 14;$$

$$\mu_9 = [0, 0, 0, 0, 3] + [1, 1, 0, 0, 1], \dim(\mu_9) = 4368, \deg(\mu_9) = 15;$$

$$\mu_{10} = [0, 2, 0, 0, 0] + [1, 0, 0, 0, 2], \dim(\mu_{10}) = 1820, \deg(\mu_{10}) = 16;$$

$$\mu_{11} = [0, 1, 0, 0, 1], \dim(\mu_{11}) = 560, \deg(\mu_{11}) = 17;$$

$$\mu_{12} = [0, 0, 1, 0, 0], \dim(\mu_{12}) = 120, \deg(\mu_{12}) = 18;$$

$$\mu_{13} = [0, 0, 0, 1, 0], \dim(\mu_{13}) = 16, \deg(\mu_{13}) = 19;$$

$$\mu_{14} = [0, 0, 0, 0, 0], \dim(\mu_{14}) = 1, \deg(\mu_{14}) = 20.$$

- D=10+0, n=3

$$\mu_0 = [0, 1, 0, 0, 0], \dim(\mu_0) = 45, \deg(\mu_0) = 8;$$

$$\mu_1 = [0, 0, 0, 0, 1] + [1, 0, 0, 1, 0], \dim(\mu_1) = 160, \deg(\mu_1) = 9;$$

$$\mu_2 = \mu_2' + \mu_2'',$$

$$\mu_2' = [0, 0, 0, 2, 0] + [1, 0, 0, 0, 0], \dim(\mu_2') = 136, \deg(\mu_2') = 10;$$

$$\mu_2'' = [1, 0, 0, 0, 1], \dim(\mu_2'') = 144, \deg(\mu_2'') = 11;$$

$$\mu_3 = [0, 0, 0, 0, 0] + [0, 0, 0, 1, 1] + [0, 1, 0, 0, 0] + [2, 0, 0, 0, 0], \dim(\mu_3) = 310, \deg(\mu_3) = 12;$$

$$\mu_4 = [0, 0, 0, 0, 1] + [1, 0, 0, 1, 0], \dim(\mu_4) = 160, \deg(\mu_4) = 13;$$

$$\mu_5 = [0, 0, 0, 1, 0], \dim(\mu_5) = 16, \deg(\mu_5) = 15;$$

$$\mu_6 = [0, 0, 0, 0, 0], \dim(\mu_6) = 1, \deg(\mu_6) = 16.$$

- D=10+0, n=4

$$\mu_0 = [1, 0, 0, 0, 0], \dim(\mu_0) = 10, \deg(\mu_0) = 10;$$

$$\mu_1 = \mu_1' + \mu_1'',$$

$$\mu_1' = [0, 0, 0, 1, 0], \dim(\mu_1') = 16, \deg(\mu_1') = 11;$$

$$\mu_1'' = [2, 0, 0, 0, 0], \dim(\mu_1'') = 54, \deg(\mu_1'') = 12;$$

$$\mu_2 = [0, 0, 0, 0, 1] + [1, 0, 0, 1, 0], \dim(\mu_2) = 160, \deg(\mu_2) = 13;$$

$$\mu_3 = [0, 0, 1, 0, 0] + [1, 0, 0, 0, 0], \dim(\mu_3) = 130, \deg(\mu_3) = 14;$$

$$\mu_4 = [0, 0, 0, 0, 0] + [0, 1, 0, 0, 0], \dim(\mu_4) = 46, \deg(\mu_4) = 16;$$

$$\mu_5 = [0, 0, 0, 0, 1], \dim(\mu_5) = 16, \deg(\mu_5) = 17.$$

- D=10+0, n=5

$$\mu_0 = [0, 0, 0, 0, 0], \dim(\mu_0) = 1, \deg(\mu_0) = 12;$$

$$\mu_1 = [1, 0, 0, 0, 0], \dim(\mu_1) = 10, \deg(\mu_1) = 14;$$

$$\mu_2 = [0, 0, 0, 1, 0], \dim(\mu_2) = 16, \deg(\mu_2) = 15;$$

$$\mu_3 = [0, 0, 0, 0, 1], \dim(\mu_3) = 16, \deg(\mu_3) = 17;$$

$$\mu_4 = [1, 0, 0, 0, 0], \dim(\mu_4) = 10, \deg(\mu_4) = 18;$$

$$\mu_5 = [0, 0, 0, 0, 0], \dim(\mu_5) = 1, \deg(\mu_5) = 20.$$

- D=9+1, n=0

$$\mu_0 = [0, 0, 0, 0], \dim(\mu_0) = 1, \deg(\mu_0) = 0;$$

$$\mu_1 = [1, 0, 0, 0], \dim(\mu_1) = 9, \deg(\mu_1) = 2;$$

$$\mu_2 = [0, 0, 1, 0] + [0, 1, 0, 0], \dim(\mu_2) = 120, \deg(\mu_2) = 4;$$

$$\mu_3 = [0, 0, 0, 1] + [0, 1, 0, 1] + [1, 0, 0, 1], \dim(\mu_3) = 576, \deg(\mu_3) = 5;$$

$$\begin{aligned} \mu_4 = & [0, 0, 0, 0] + [0, 0, 0, 2] + [0, 2, 0, 0] + [1, 0, 0, 0] + [1, 0, 0, 2] + [1, 1, 0, 0] + \\ & + [2, 0, 0, 0], \dim(\mu_4) = 1830, \deg(\mu_4) = 6; \end{aligned}$$

$$\mu_5 = \mu_5' + \mu_5'',$$

$$\mu_5' = [0, 0, 0, 3] + [0, 1, 0, 1] + [1, 0, 0, 1] + [1, 1, 0, 1] + [2, 0, 0, 1], \dim(\mu_5') = 4368, \deg(\mu_5') = 7;$$

$$\mu_5'' = [0, 0, 0, 0], \dim(\mu_5'') = 1, \deg(\mu_5'') = 8;$$

$$\begin{aligned} \mu_6 = & [0, 0, 1, 0] + [0, 1, 0, 0] + [0, 1, 0, 2] + [1, 0, 0, 2] + [1, 0, 1, 0] + [1, 1, 0, 0] + \\ & + [2, 0, 1, 0] + [2, 1, 0, 0], \dim(\mu_6) = 8008, \deg(\mu_6) = 8; \end{aligned}$$

$$\begin{aligned} \mu_7 = & [0, 0, 0, 1] + [0, 0, 1, 1] + [0, 1, 0, 1] + [1, 0, 0, 1] + [1, 0, 1, 1] + [1, 1, 0, 1] + \\ & + [2, 0, 0, 1] + [3, 0, 0, 1], \dim(\mu_7) = 11440, \deg(\mu_7) = 9; \end{aligned}$$

$$\begin{aligned} \mu_8 = & [0, 0, 0, 0] + [0, 0, 0, 2] + [0, 0, 1, 0] + [0, 0, 2, 0] + [0, 1, 1, 0] + [0, 2, 0, 0] + \\ & + [1, 0, 0, 0] + [1, 0, 0, 2] + [1, 0, 1, 0] + [2, 0, 0, 0] + [2, 0, 0, 2] + [2, 0, 1, 0] + \\ & + [3, 0, 0, 0] + [4, 0, 0, 0], \dim(\mu_8) = 12870, \deg(\mu_8) = 10; \end{aligned}$$

$$\begin{aligned}
\mu_9 &= [0, 0, 0, 1] + [0, 0, 1, 1] + [0, 1, 0, 1] + [1, 0, 0, 1] + [1, 0, 1, 1] + [1, 1, 0, 1] + \\
&\quad + [2, 0, 0, 1] + [3, 0, 0, 1], \dim(\mu_9) = 11440, \deg(\mu_9) = 11; \\
\mu_{10} &= [0, 0, 1, 0] + [0, 1, 0, 0] + [0, 1, 0, 2] + [1, 0, 0, 2] + [1, 0, 1, 0] + [1, 1, 0, 0] + \\
&\quad + [2, 0, 1, 0] + [2, 1, 0, 0], \dim(\mu_{10}) = 8008, \deg(\mu_{10}) = 12; \\
\mu_{11} &= [0, 0, 0, 3] + [0, 1, 0, 1] + [1, 0, 0, 1] + [1, 1, 0, 1] + [2, 0, 0, 1], \dim(\mu_{11}) = 4368, \deg(\mu_{11}) = 13; \\
\mu_{12} &= [0, 0, 0, 2] + [0, 2, 0, 0] + [1, 0, 0, 2] + [1, 1, 0, 0] + [2, 0, 0, 0], \dim(\mu_{12}) = 1820, \deg(\mu_{12}) = 14; \\
\mu_{13} &= [0, 1, 0, 1] + [1, 0, 0, 1], \dim(\mu_{13}) = 560, \deg(\mu_{13}) = 15; \\
\mu_{14} &= [0, 0, 1, 0] + [0, 1, 0, 0], \dim(\mu_{14}) = 120, \deg(\mu_{14}) = 16; \\
\mu_{15} &= [0, 0, 0, 1], \dim(\mu_{15}) = 16, \deg(\mu_{15}) = 17; \\
\mu_{16} &= [0, 0, 0, 0], \dim(\mu_{16}) = 1, \deg(\mu_{16}) = 18.
\end{aligned}$$

• D=9+1, n=1

$$\begin{aligned}
\mu_0 &= [0, 0, 1, 0], \dim(\mu_0) = 84, \deg(\mu_0) = 4; \\
\mu_1 &= [0, 0, 0, 1] + [0, 1, 0, 1] + [1, 0, 0, 1], \dim(\mu_1) = 576, \deg(\mu_1) = 5; \\
\mu_2 &= [0, 0, 0, 0] + [0, 0, 0, 2] + [0, 0, 1, 0] + [0, 1, 0, 0] + [0, 2, 0, 0] + [1, 0, 0, 0] + \\
&\quad + [1, 0, 0, 2] + [1, 1, 0, 0] + [2, 0, 0, 0], \dim(\mu_2) = 1950, \deg(\mu_2) = 6; \\
\mu_3 &= [0, 0, 0, 1] + [0, 0, 0, 3] + [0, 1, 0, 1] + 2 \times [1, 0, 0, 1] + [1, 1, 0, 1] + [2, 0, 0, 1], \\
&\quad \dim(\mu_3) = 4512, \deg(\mu_3) = 7; \\
\mu_4 &= \mu_4' + \mu_4'', \\
\mu_4' &= [0, 0, 1, 0] + [0, 1, 0, 0] + [0, 1, 0, 2] + [1, 0, 0, 2] + [1, 0, 1, 0] + [1, 1, 0, 0] + \\
&\quad + [2, 0, 0, 0] + [2, 0, 1, 0] + [2, 1, 0, 0], \dim(\mu_4') = 8052, \deg(\mu_4') = 8; \\
\mu_4'' &= [0, 0, 0, 1], \dim(\mu_4'') = 16, \deg(\mu_4'') = 9; \\
\mu_5 &= \mu_5' + \mu_5'', \\
\mu_5' &= [0, 0, 0, 1] + [0, 0, 1, 1] + [0, 1, 0, 1] + [1, 0, 0, 1] + [1, 0, 1, 1] + [1, 1, 0, 1] + \\
&\quad + [2, 0, 0, 1] + [3, 0, 0, 1], \dim(\mu_5') = 11440, \deg(\mu_5') = 9; \\
\mu_5'' &= [1, 0, 0, 0], \dim(\mu_5'') = 9, \deg(\mu_5'') = 10;
\end{aligned}$$

$$\begin{aligned}
\mu_6 &= [0, 0, 0, 0] + [0, 0, 0, 2] + [0, 0, 1, 0] + [0, 0, 2, 0] + [0, 1, 1, 0] + [0, 2, 0, 0] + \\
&\quad + [1, 0, 0, 0] + [1, 0, 0, 2] + [1, 0, 1, 0] + [2, 0, 0, 0] + [2, 0, 0, 2] + [2, 0, 1, 0] + \\
&\quad + [3, 0, 0, 0] + [4, 0, 0, 0], \dim(\mu_6) = 12870, \deg(\mu_6) = 10; \\
\mu_7 &= [0, 0, 0, 1] + [0, 0, 1, 1] + [0, 1, 0, 1] + [1, 0, 0, 1] + [1, 0, 1, 1] + [1, 1, 0, 1] + \\
&\quad + [2, 0, 0, 1] + [3, 0, 0, 1], \dim(\mu_7) = 11440, \deg(\mu_7) = 11; \\
\mu_8 &= [0, 0, 1, 0] + [0, 1, 0, 0] + [0, 1, 0, 2] + [1, 0, 0, 2] + [1, 0, 1, 0] + [1, 1, 0, 0] + \\
&\quad + [2, 0, 1, 0] + [2, 1, 0, 0], \dim(\mu_8) = 8008, \deg(\mu_8) = 12; \\
\mu_9 &= [0, 0, 0, 3] + [0, 1, 0, 1] + [1, 0, 0, 1] + [1, 1, 0, 1] + [2, 0, 0, 1], \dim(\mu_9) = 4368, \deg(\mu_9) = 13; \\
\mu_{10} &= [0, 0, 0, 2] + [0, 2, 0, 0] + [1, 0, 0, 2] + [1, 1, 0, 0] + [2, 0, 0, 0], \dim(\mu_{10}) = 1820, \deg(\mu_{10}) = 14; \\
\mu_{11} &= [0, 1, 0, 1] + [1, 0, 0, 1], \dim(\mu_{11}) = 560, \deg(\mu_{11}) = 15; \\
\mu_{12} &= [0, 0, 1, 0] + [0, 1, 0, 0], \dim(\mu_{12}) = 120, \deg(\mu_{12}) = 16; \\
\mu_{13} &= [0, 0, 0, 1], \dim(\mu_{13}) = 16, \deg(\mu_{13}) = 17; \\
\mu_{14} &= [0, 0, 0, 0], \dim(\mu_{14}) = 1, \deg(\mu_{14}) = 18.
\end{aligned}$$

- D=9+1, n=2

$$\begin{aligned}
\mu_0 &= [0, 1, 0, 0], \dim(\mu_0) = 36, \deg(\mu_0) = 6; \\
\mu_1 &= [0, 0, 0, 1] + [1, 0, 0, 1], \dim(\mu_1) = 144, \deg(\mu_1) = 7; \\
\mu_2 &= [0, 0, 0, 0] + [0, 0, 0, 2] + [1, 0, 0, 0] + [2, 0, 0, 0], \dim(\mu_2) = 180, \deg(\mu_2) = 8; \\
\mu_3 &= [0, 0, 0, 0] + [0, 0, 0, 2] + [1, 0, 0, 0] + [2, 0, 0, 0], \dim(\mu_3) = 180, \deg(\mu_3) = 10; \\
\mu_4 &= [0, 0, 0, 1] + [1, 0, 0, 1], \dim(\mu_4) = 144, \deg(\mu_4) = 11; \\
\mu_5 &= [0, 1, 0, 0], \dim(\mu_5) = 36, \deg(\mu_5) = 12.
\end{aligned}$$

- D=9+1, n=3

$$\mu_0 = [1, 0, 0, 0], \dim(\mu_0) = 9, \deg(\mu_0) = 8;$$

$$\mu_1 = \mu_1' + \mu_1'',$$

$$\mu_1' = [0, 0, 0, 1], \dim(\mu_1') = 16, \deg(\mu_1') = 9;$$

$$\mu_1'' = [2, 0, 0, 0], \dim(\mu_1'') = 44, \deg(\mu_1'') = 10;$$

$$\mu_2 = [0, 0, 0, 1] + [1, 0, 0, 1], \dim(\mu_2) = 144, \deg(\mu_2) = 11;$$

$$\mu_3 = [0, 0, 0, 0] + [0, 0, 1, 0] + [0, 1, 0, 0] + [1, 0, 0, 0], \dim(\mu_3) = 130, \deg(\mu_3) = 12;$$

$$\mu_4 = \mu_4' + \mu_4'',$$

$$\mu_4' = [0, 0, 0, 1], \dim(\mu_4') = 16, \deg(\mu_4') = 13;$$

$$\mu_4'' = [0, 1, 0, 0], \dim(\mu_4'') = 36, \deg(\mu_4'') = 14;$$

$$\mu_5 = [0, 0, 0, 1], \dim(\mu_5) = 16, \deg(\mu_5) = 15;$$

$$\mu_6 = [0, 0, 0, 0], \dim(\mu_6) = 1, \deg(\mu_6) = 16.$$

- D=9+1, n=4

$$\mu_0 = [0, 0, 0, 0], \dim(\mu_0) = 1, \deg(\mu_0) = 10;$$

$$\mu_1 = [0, 0, 0, 0] + [1, 0, 0, 0], \dim(\mu_1) = 10, \deg(\mu_1) = 12;$$

$$\mu_2 = [0, 0, 0, 1], \dim(\mu_2) = 16, \deg(\mu_2) = 13;$$

$$\mu_3 = [0, 0, 0, 1], \dim(\mu_3) = 16, \deg(\mu_3) = 15;$$

$$\mu_4 = [0, 0, 0, 0] + [1, 0, 0, 0], \dim(\mu_4) = 10, \deg(\mu_4) = 16;$$

$$\mu_5 = [0, 0, 0, 0], \dim(\mu_5) = 1, \deg(\mu_5) = 18.$$

- D=8+2, n=0

$$\mu_0 = [0, 0, 0, 0, 0], \dim(\mu_0) = 1, \deg(\mu_0) = 0;$$

$$\mu_1 = [1, 0, 0, 0, 0], \dim(\mu_1) = 8, \deg(\mu_1) = 2;$$

$$\mu_2 = [0, 1, 0, 0, 0], \dim(\mu_2) = 56, \deg(\mu_2) = 4;$$

$$\mu_3 = [0, 0, 0, 1, -1] + [0, 0, 1, 0, 1] + [1, 0, 0, 1, 1] + [1, 0, 1, 0, -1], \dim(\mu_3) = 128, \deg(\mu_3) = 5;$$

$$\begin{aligned} \mu_4 = & [0, 0, 0, 0, -2] + [0, 0, 0, 0, 2] + [0, 0, 0, 2, 2] + [0, 0, 2, 0, -2] + [1, 0, 0, 0, 0] + \\ & + [2, 0, 0, 0, -2] + [2, 0, 0, 0, 2], \dim(\mu_4) = 150, \deg(\mu_4) = 6; \end{aligned}$$



$$\mu_5 = \mu_5' + \mu_5'',$$

$$\mu_5' = [1, 0, 0, 1, 3] + [1, 0, 1, 0, -3], \dim(\mu_5') = 112, \deg(\mu_5') = 7;$$

$$\mu_5'' = [0, 0, 0, 0, 0], \dim(\mu_5'') = 1, \deg(\mu_5'') = 8;$$

$$\mu_6 = [0, 1, 0, 0, -4] + [0, 1, 0, 0, 4], \dim(\mu_6) = 56, \deg(\mu_6) = 8;$$

$$\mu_7 = [0, 0, 0, 1, -5] + [0, 0, 1, 0, 5], \dim(\mu_7) = 16, \deg(\mu_7) = 9;$$

$$\mu_8 = [0, 0, 0, 0, -6] + [0, 0, 0, 0, 6], \dim(\mu_8) = 2, \deg(\mu_8) = 10.$$

- D=8+2, n=1

$$\mu_0 = [0, 1, 0, 0, 0], \dim(\mu_0) = 28, \deg(\mu_0) = 4;$$

$$\mu_1 = [0, 0, 0, 1, -1] + [0, 0, 1, 0, 1] + [1, 0, 0, 1, 1] + [1, 0, 1, 0, -1], \dim(\mu_1) = 128, \deg(\mu_1) = 5;$$

$$\begin{aligned} \mu_2 = & [0, 0, 0, 0, -2] + [0, 0, 0, 0, 2] + [0, 0, 0, 2, 2] + [0, 0, 1, 1, 0] + [0, 0, 2, 0, -2] + \\ & + 2 \times [1, 0, 0, 0, 0] + [2, 0, 0, 0, -2] + [2, 0, 0, 0, 2], \dim(\mu_2) = 214, \deg(\mu_2) = 6; \end{aligned}$$

$$\mu_3 = \mu_3' + \mu_3'',$$

$$\mu_3' = [0, 0, 0, 1, 1] + [0, 0, 1, 0, -1] + [1, 0, 0, 1, 3] + [1, 0, 1, 0, -3], \dim(\mu_3') = 128, \deg(\mu_3') = 7;$$

$$\mu_3'' = [0, 0, 0, 0, 0] + [0, 0, 0, 2, 0] + [0, 0, 2, 0, 0] + [2, 0, 0, 0, 0], \dim(\mu_3'') = 106, \deg(\mu_3'') = 8;$$

$$\mu_4 = \mu_4' + \mu_4'',$$

$$\mu_4' = [0, 1, 0, 0, -4] + [0, 1, 0, 0, 4], \dim(\mu_4') = 56, \deg(\mu_4') = 8;$$

$$\mu_4'' = [0, 0, 0, 1, -1] + [0, 0, 1, 0, 1] + [1, 0, 0, 1, 1] + [1, 0, 1, 0, -1], \dim(\mu_4'') = 128, \deg(\mu_4'') = 9;$$

$$\mu_5 = \mu_5' + \mu_5'',$$

$$\mu_5' = [0, 0, 0, 1, -5] + [0, 0, 1, 0, 5], \dim(\mu_5') = 16, \deg(\mu_5') = 9;$$

$$\mu_5'' = [0, 1, 0, 0, -2] + [0, 1, 0, 0, 2] + [1, 0, 0, 0, 0], \dim(\mu_5'') = 64, \deg(\mu_5'') = 10;$$

$$\mu_6 = \mu_6' + \mu_6'',$$

$$\mu_6' = [0, 0, 0, 0, -6] + [0, 0, 0, 0, 6], \dim(\mu_6') = 2, \deg(\mu_6') = 10;$$

$$\mu_6'' = [0, 0, 0, 1, -3] + [0, 0, 1, 0, 3], \dim(\mu_6'') = 16, \deg(\mu_6'') = 11;$$

$$\mu_7 = [0, 0, 0, 0, -4] + [0, 0, 0, 0, 4], \dim(\mu_7) = 2, \deg(\mu_7) = 12.$$

- D=8+2, n=2

$$\mu_0 = [1, 0, 0, 0, 0], \dim(\mu_0) = 8, \deg(\mu_0) = 6;$$

$$\mu_1 = \mu_1' + \mu_1'',$$

$$\mu_1' = [0, 0, 0, 1, 1] + [0, 0, 1, 0, -1], \dim(\mu_1') = 16, \deg(\mu_1') = 7;$$

$$\mu_1'' = [2, 0, 0, 0, 0], \dim(\mu_1'') = 35, \deg(\mu_1'') = 8;$$

$$\mu_2 = \mu_2' + \mu_2'',$$

$$\mu_2' = [0, 0, 0, 0, 0], \dim(\mu_2') = 1, \deg(\mu_2') = 8;$$

$$\mu_2'' = [0, 0, 0, 1, -1] + [0, 0, 1, 0, 1] + [1, 0, 0, 1, 1] + [1, 0, 1, 0, -1], \dim(\mu_2'') = 128, \deg(\mu_2'') = 9;$$

$$\mu_3 = [0, 0, 0, 0, -2] + [0, 0, 0, 0, 2] + [0, 0, 1, 1, 0] + [0, 1, 0, 0, -2] + [0, 1, 0, 0, 2] + \\ + 2 \times [1, 0, 0, 0, 0], \dim(\mu_3) = 130, \deg(\mu_3) = 10;$$

$$\mu_4 = \mu_4' + \mu_4'',$$

$$\mu_4' = [0, 0, 0, 1, -3] + [0, 0, 0, 1, 1] + [0, 0, 1, 0, -1] + [0, 0, 1, 0, 3], \dim(\mu_4') = 32, \deg(\mu_4') = 11;$$

$$\mu_4'' = [0, 1, 0, 0, 0], \dim(\mu_4'') = 28, \deg(\mu_4'') = 12;$$

$$\mu_5 = \mu_5' + \mu_5'',$$

$$\mu_5' = [0, 0, 0, 0, -4] + [0, 0, 0, 0, 4], \dim(\mu_5') = 2, \deg(\mu_5') = 12;$$

$$\mu_5'' = [0, 0, 0, 1, -1] + [0, 0, 1, 0, 1], \dim(\mu_5'') = 16, \deg(\mu_5'') = 13;$$

$$\mu_6 = [0, 0, 0, 0, -2] + [0, 0, 0, 0, 2], \dim(\mu_6) = 2, \deg(\mu_6) = 14.$$

- D=8+2, n=3

$$\mu_0 = [0, 0, 0, 0, 0], \dim(\mu_0) = 1, \deg(\mu_0) = 8;$$

$$\mu_1 = [0, 0, 0, 0, -2] + [0, 0, 0, 0, 2] + [1, 0, 0, 0, 0], \dim(\mu_1) = 10, \deg(\mu_1) = 10;$$

$$\mu_2 = [0, 0, 0, 1, 1] + [0, 0, 1, 0, -1], \dim(\mu_2) = 16, \deg(\mu_2) = 11;$$

$$\mu_3 = [0, 0, 0, 1, -1] + [0, 0, 1, 0, 1], \dim(\mu_3) = 16, \deg(\mu_3) = 13;$$

$$\mu_4 = [0, 0, 0, 0, -2] + [0, 0, 0, 0, 2] + [1, 0, 0, 0, 0], \dim(\mu_4) = 10, \deg(\mu_4) = 14;$$

$$\mu_5 = [0, 0, 0, 0, 0], \dim(\mu_5) = 1, \deg(\mu_5) = 16.$$

- D=7+3, n=0

$$\mu_0 = [0, 0, 0, 0], \dim(\mu_0) = 1, \deg(\mu_0) = 0;$$

$$\mu_1 = [1, 0, 0, 0], \dim(\mu_1) = 7, \deg(\mu_1) = 2;$$

$$\mu_2 = [0, 1, 0, 0] + [1, 0, 0, 0], \dim(\mu_2) = 28, \deg(\mu_2) = 4;$$

$$\mu_3 = \mu_3' + \mu_3'',$$

$$\mu_3' = [0, 0, 1, 1], \dim(\mu_3') = 16, \deg(\mu_3') = 5;$$

$$\mu_3'' = [0, 0, 0, 0] + [0, 0, 2, 0] + [2, 0, 0, 0], \dim(\mu_3'') = 63, \deg(\mu_3'') = 6;$$

$$\mu_4 = \mu_4' + \mu_4'',$$

$$\mu_4' = [0, 0, 0, 2], \dim(\mu_4') = 3, \deg(\mu_4') = 6;$$

$$\mu_4'' = [0, 0, 1, 1] + [1, 0, 1, 1], \dim(\mu_4'') = 112, \deg(\mu_4'') = 7;$$

$$\mu_5 = [0, 0, 0, 0] + [0, 1, 0, 2] + [1, 0, 0, 2], \dim(\mu_5) = 85, \deg(\mu_5) = 8;$$

$$\mu_6 = [0, 0, 1, 3], \dim(\mu_6) = 32, \deg(\mu_6) = 9;$$

$$\mu_7 = [0, 0, 0, 4], \dim(\mu_7) = 5, \deg(\mu_7) = 10.$$

- D=7+3, n=1

$$\mu_0 = [1, 0, 0, 0], \dim(\mu_0) = 7, \deg(\mu_0) = 4;$$

$$\mu_1 = \mu_1' + \mu_1'',$$

$$\mu_1' = [0, 0, 1, 1], \dim(\mu_1') = 16, \deg(\mu_1') = 5;$$

$$\mu_1'' = [2, 0, 0, 0], \dim(\mu_1'') = 27, \deg(\mu_1'') = 6;$$

$$\mu_2 = \mu_2' + \mu_2'',$$

$$\mu_2' = [0, 0, 0, 2], \dim(\mu_2') = 3, \deg(\mu_2') = 6;$$

$$\mu_2'' = [0, 0, 1, 1] + [1, 0, 1, 1], \dim(\mu_2'') = 112, \deg(\mu_2'') = 7;$$

$$\mu_3 = [0, 0, 0, 0] + [0, 0, 0, 2] + [0, 0, 2, 0] + [0, 1, 0, 2] + [1, 0, 0, 0] + [1, 0, 0, 2],$$

$$\dim(\mu_3) = 130, \deg(\mu_3) = 8;$$

$$\mu_4 = \mu_4' + \mu_4'',$$

$$\mu_4' = [0, 0, 1, 1] + [0, 0, 1, 3], \dim(\mu_4') = 48, \deg(\mu_4') = 9;$$

$$\mu_4'' = [0, 1, 0, 0], \dim(\mu_4'') = 21, \deg(\mu_4'') = 10;$$

$$\mu_5 = \mu_5' + \mu_5'',$$

$$\mu_5' = [0, 0, 0, 4], \dim(\mu_5') = 5, \deg(\mu_5') = 10;$$

$$\mu_5'' = [0, 0, 1, 1], \dim(\mu_5'') = 16, \deg(\mu_5'') = 11;$$

$$\mu_6 = [0, 0, 0, 2], \dim(\mu_6) = 3, \deg(\mu_6) = 12.$$

- D=7+3, n=2

$$\mu_0 = [0, 0, 0, 0], \dim(\mu_0) = 1, \deg(\mu_0) = 6;$$

$$\mu_1 = [0, 0, 0, 2] + [1, 0, 0, 0], \dim(\mu_1) = 10, \deg(\mu_1) = 8;$$

$$\mu_2 = [0, 0, 1, 1], \dim(\mu_2) = 16, \deg(\mu_2) = 9;$$

$$\mu_3 = [0, 0, 1, 1], \dim(\mu_3) = 16, \deg(\mu_3) = 11;$$

$$\mu_4 = [0, 0, 0, 2] + [1, 0, 0, 0], \dim(\mu_4) = 10, \deg(\mu_4) = 12;$$

$$\mu_5 = [0, 0, 0, 0], \dim(\mu_5) = 1, \deg(\mu_5) = 14.$$

- D=6+4, n=0

$$\mu_0 = [0, 0, 0, 0, 0], \dim(\mu_0) = 1, \deg(\mu_0) = 0;$$

$$\mu_1 = [0, 1, 0, 0, 0], \dim(\mu_1) = 6, \deg(\mu_1) = 2;$$

$$\mu_2 = [0, 0, 0, 0, 0] + [1, 0, 1, 0, 0], \dim(\mu_2) = 16, \deg(\mu_2) = 4;$$

$$\mu_3 = [0, 0, 0, 1, 1] + [0, 0, 2, 0, 0] + [0, 1, 0, 0, 0] + [2, 0, 0, 0, 0], \dim(\mu_3) = 30, \deg(\mu_3) = 6;$$

$$\mu_4 = \mu_4' + \mu_4'',$$

$$\mu_4' = [0, 0, 1, 1, 0] + [1, 0, 0, 0, 1], \dim(\mu_4') = 16, \deg(\mu_4') = 7;$$

$$\mu_4'' = [1, 0, 1, 0, 0], \dim(\mu_4'') = 15, \deg(\mu_4'') = 8;$$

$$\mu_5 = [0, 0, 1, 0, 1] + [1, 0, 0, 1, 0], \dim(\mu_5) = 16, \deg(\mu_5) = 9;$$

$$\mu_6 = [0, 0, 0, 1, 1], \dim(\mu_6) = 4, \deg(\mu_6) = 10.$$

- D=6+4, n=1

$$\begin{aligned}
\mu_0 &= [0, 0, 0, 0, 0], \dim(\mu_0) = 1, \deg(\mu_0) = 4; \\
\mu_1 &= [0, 0, 0, 1, 1] + [0, 1, 0, 0, 0], \dim(\mu_1) = 10, \deg(\mu_1) = 6; \\
\mu_2 &= [0, 0, 1, 1, 0] + [1, 0, 0, 0, 1], \dim(\mu_2) = 16, \deg(\mu_2) = 7; \\
\mu_3 &= [0, 0, 1, 0, 1] + [1, 0, 0, 1, 0], \dim(\mu_3) = 16, \deg(\mu_3) = 9; \\
\mu_4 &= [0, 0, 0, 1, 1] + [0, 1, 0, 0, 0], \dim(\mu_4) = 10, \deg(\mu_4) = 10; \\
\mu_5 &= [0, 0, 0, 0, 0], \dim(\mu_5) = 1, \deg(\mu_5) = 12.
\end{aligned}$$

- D=5+5, n=0

$$\begin{aligned}
\mu_0 &= [0, 0, 0, 0], \dim(\mu_0) = 1, \deg(\mu_0) = 0; \\
\mu_1 &= [1, 0, 0, 0], \dim(\mu_1) = 5, \deg(\mu_1) = 2; \\
\mu_2 &= [0, 2, 0, 0], \dim(\mu_2) = 10, \deg(\mu_2) = 4; \\
\mu_3 &= [0, 2, 0, 0], \dim(\mu_3) = 10, \deg(\mu_3) = 6; \\
\mu_4 &= [1, 0, 0, 0], \dim(\mu_4) = 5, \deg(\mu_4) = 8; \\
\mu_5 &= [0, 0, 0, 0], \dim(\mu_5) = 1, \deg(\mu_5) = 10.
\end{aligned}$$

- D=4+6, n=0

$$\begin{aligned}
\mu_0 &= [0, 0, 0, 0, 0], \dim(\mu_0) = 1, \deg(\mu_0) = 0; \\
\mu_1 &= [1, 1, 0, 0, 0], \dim(\mu_1) = 4, \deg(\mu_1) = 2; \\
\mu_2 &= [0, 2, 0, 0, 0] + [2, 0, 0, 0, 0], \dim(\mu_2) = 6, \deg(\mu_2) = 4; \\
\mu_3 &= [1, 1, 0, 0, 0], \dim(\mu_3) = 4, \deg(\mu_3) = 6; \\
\mu_4 &= [0, 0, 0, 0, 0], \dim(\mu_4) = 1, \deg(\mu_4) = 8.
\end{aligned}$$

- D=3+7, n=0

$$\mu_0 = [0, 0, 0, 0], \dim(\mu_0) = 1, \deg(\mu_0) = 0;$$

$$\mu_1 = [2, 0, 0, 0], \dim(\mu_1) = 3, \deg(\mu_1) = 2;$$

$$\mu_2 = [2, 0, 0, 0], \dim(\mu_2) = 3, \deg(\mu_2) = 4;$$

$$\mu_3 = [0, 0, 0, 0], \dim(\mu_3) = 1, \deg(\mu_3) = 6.$$

- D=2+8, n=0

$$\mu_0 = [0, 0, 0, 0, 0], \dim(\mu_0) = 1, \deg(\mu_0) = 0;$$

$$\mu_1 = [0, 0, 0, 0, -2] + [0, 0, 0, 0, 2], \dim(\mu_1) = 2, \deg(\mu_1) = 2;$$

$$\mu_2 = [0, 0, 0, 0, 0], \dim(\mu_2) = 1, \deg(\mu_2) = 4.$$

- D=1+9, n=0

$$\mu_0 = [0, 0, 0, 0], \dim(\mu_0) = 1, \deg(\mu_0) = 0;$$

$$\mu_1 = [0, 0, 0, 0], \dim(\mu_1) = 1, \deg(\mu_1) = 2.$$

## A.2 Resolution of the cohomology modules of 6D Lie algebra of supersymmetries

We now give the structure of  $\mu_i$  as *aut*-module and its grading in the case of six-dimensional Lie algebra of supersymmetries.

- D=6+0, n=0

$$\mu_0 = [0, 0, 0, 0], \dim(\mu_0) = 1, \deg(\mu_0) = 0;$$

$$\mu_1 = [0, 1, 0, 0], \dim(\mu_1) = 6, \deg(\mu_1) = 2;$$

$$\mu_2 = [1, 0, 0, 1], \dim(\mu_2) = 8, \deg(\mu_2) = 3;$$

$$\mu_3 = [0, 0, 0, 2], \dim(\mu_3) = 3, \deg(\mu_3) = 4.$$

- D=6+0, n=1

$$\mu_0 = [1, 0, 0, 1], \dim(\mu_0) = 8, \deg(\mu_0) = 3;$$

$$\mu_1 = [0, 0, 0, 2] + [1, 0, 1, 0], \dim(\mu_1) = 18, \deg(\mu_1) = 4;$$

$$\mu_2 = [0, 0, 2, 0] + [0, 1, 0, 2] + [2, 0, 0, 0], \dim(\mu_2) = 38, \deg(\mu_2) = 6;$$

$$\mu_3 = [0, 1, 1, 1] + [1, 0, 0, 1] + [1, 0, 0, 3], \dim(\mu_3) = 64, \deg(\mu_3) = 7;$$

$$\mu_4 = [0, 0, 0, 0] + [0, 0, 0, 4] + [0, 2, 0, 0] + [1, 0, 1, 2], \dim(\mu_4) = 71, \deg(\mu_4') = 8;$$

$$\mu_5 = [0, 0, 1, 3] + [1, 1, 0, 1], \dim(\mu_5) = 56, \deg(\mu_5) = 9;$$

$$\mu_6 = [0, 1, 0, 2] + [2, 0, 0, 0], \dim(\mu_6) = 28, \deg(\mu_6) = 10;$$

$$\mu_7 = [1, 0, 0, 1], \dim(\mu_7) = 8, \deg(\mu_7) = 11;$$

$$\mu_8 = [0, 0, 0, 0], \dim(\mu_8) = 1, \deg(\mu_8) = 12.$$

- D=6+0, n=2

$$\mu_0 = [0, 1, 0, 2], \dim(\mu_0) = 18, \deg(\mu_0) = 6;$$

$$\mu_1 = [0, 1, 1, 1] + [1, 0, 0, 1] + [1, 0, 0, 3], \dim(\mu_1) = 64, \deg(\mu_1) = 7;$$

$$\mu_2 = [0, 0, 0, 0] + [0, 0, 0, 2] + [0, 0, 0, 4] + [0, 2, 0, 0] + [1, 0, 1, 0] + [1, 0, 1, 2], \dim(\mu_2) = 89, \deg(\mu_2) = 8;$$

$$\mu_3 = [0, 0, 1, 1] + [0, 0, 1, 3] + [1, 1, 0, 1], \dim(\mu_3) = 64, \deg(\mu_3) = 9;$$

$$\mu_4 = [0, 1, 0, 2] + [2, 0, 0, 0], \dim(\mu_4) = 28, \deg(\mu_4') = 10;$$

$$\mu_5 = [1, 0, 0, 1], \dim(\mu_5) = 8, \deg(\mu_5) = 11;$$

$$\mu_6 = [0, 0, 0, 0], \dim(\mu_6) = 1, \deg(\mu_6) = 12.$$

- D=6+0, n=3

$$\mu_0 = [0, 0, 0, 2], \dim(\mu_0) = 3, \deg(\mu_0) = 8;$$

$$\mu_1 = [0, 0, 1, 1], \dim(\mu_1) = 8, \deg(\mu_1) = 9;$$

$$\mu_2 = [0, 1, 0, 0], \dim(\mu_2) = 6, \deg(\mu_2) = 10;$$

$$\mu_3 = [0, 0, 0, 0], \dim(\mu_3) = 1, \deg(\mu_3) = 12.$$

### A.3 Resolution of the cohomology modules of 5D Lie algebra of supersymmetries

We now give the structure of  $\mu_i$  as *aut*-module and its grading in the case of five-dimensional Lie algebra of supersymmetries.

- D=5+0, n=0

$$\mu_0 = [0, 0, 0], \dim(\mu_0) = 1, \deg(\mu_0) = 0;$$

$$\mu_1 = [1, 0, 0], \dim(\mu_1) = 5, \deg(\mu_1) = 2;$$

$$\mu_2 = [0, 2, 0] + [1, 0, 2], \dim(\mu_2) = 25, \deg(\mu_2) = 4;$$

$$\mu_3 = [0, 1, 1] + [0, 1, 3] + [1, 1, 1], \dim(\mu_3) = 56, \deg(\mu_3) = 5;$$

$$\mu_4 = [0, 0, 0] + [0, 0, 4] + [0, 2, 2] + [1, 0, 0] + [1, 0, 2] + [2, 0, 0], \dim(\mu_4) = 70, \deg(\mu_4') = 6;$$

$$\mu_5 = [0, 1, 1] + [0, 1, 3] + [1, 1, 1], \dim(\mu_5) = 56, \deg(\mu_5) = 7;$$

$$\mu_6 = [0, 0, 2] + [0, 2, 0] + [1, 0, 2], \dim(\mu_6) = 28, \deg(\mu_6) = 8;$$

$$\mu_7 = [0, 1, 1], \dim(\mu_7) = 8, \deg(\mu_7) = 9;$$

$$\mu_8 = [0, 0, 0], \dim(\mu_8) = 1, \deg(\mu_8) = 10.$$

- D=5+0, n=1

$$\mu_0 = [1, 0, 2], \dim(\mu_0) = 15, \deg(\mu_0) = 4;$$

$$\mu_1 = [0, 1, 1] + [0, 1, 3] + [1, 1, 1], \dim(\mu_1) = 56, \deg(\mu_1) = 5;$$

$$\mu_2 = [0, 0, 0] + [0, 0, 2] + [0, 0, 4] + [0, 2, 0] + [0, 2, 2] + [1, 0, 0] + [1, 0, 2] + [2, 0, 0], \dim(\mu_2) = 83, \deg(\mu_2) = 6;$$

$$\mu_3 = 2 \times [0, 1, 1] + [0, 1, 3] + [1, 1, 1], \dim(\mu_3) = 64, \deg(\mu_3) = 7;$$

$$\mu_4 = [0, 0, 0] + [0, 0, 2] + [0, 2, 0] + [1, 0, 2], \dim(\mu_4) = 29, \deg(\mu_4') = 8;$$

$$\mu_5 = [0, 1, 1], \dim(\mu_5) = 8, \deg(\mu_5) = 9;$$

$$\mu_6 = [0, 0, 0], \dim(\mu_6) = 1, \deg(\mu_6) = 10.$$



- D=5+0, n=2

$$\mu_0 = [0, 0, 2], \dim(\mu_0) = 3, \deg(\mu_0) = 6;$$

$$\mu_1 = [0, 1, 1], \dim(\mu_1) = 8, \deg(\mu_1) = 7;$$

$$\mu_2 = [0, 0, 0] + [1, 0, 0], \dim(\mu_2) = 6, \deg(\mu_2) = 8;$$

$$\mu_3 = [0, 0, 0], \dim(\mu_3) = 1, \deg(\mu_3) = 10.$$

#### A.4 Resolution of the cohomology modules of 4D Lie algebra of supersymmetries

We now give the structure of  $\mu_i$  as *aut*-module and its grading in the case of four-dimensional Lie algebra of supersymmetries.

- D=4+0, n=0

$$\mu_0 = [0, 0], \dim(\mu_0) = 1, \deg(\mu_0) = 0;$$

$$\mu_1 = [1, 1], \dim(\mu_1) = 4, \deg(\mu_1) = 2;$$

$$\mu_2 = [0, 1] + [1, 0], \dim(\mu_2) = 4, \deg(\mu_2) = 3;$$

$$\mu_3 = [0, 0], \dim(\mu_3) = 1, \deg(\mu_3) = 4.$$

- D=4+0, n=1

$$\mu_0 = [0, 1] + [1, 0], \dim(\mu_0) = 15, \deg(\mu_0) = 4;$$

$$\mu_1 = [0, 0] + [0, 2] + [2, 0], \dim(\mu_1) = 56, \deg(\mu_1) = 5;$$

$$\mu_2 = 2 \times [0, 0] + [1, 1], \dim(\mu_2) = 83, \deg(\mu_2) = 6;$$

$$\mu_3 = [0, 1] + [1, 0], \dim(\mu_3) = 64, \deg(\mu_3) = 7;$$

$$\mu_4 = [0, 0], \dim(\mu_4) = 29, \deg(\mu_4') = 8.$$

- D=4+0, n=2

$$\mu_0 = 2 \times [0, 0], \dim(\mu_0) = 3, \deg(\mu_0) = 6;$$

$$\mu_1 = [0, 1] + [1, 0], \dim(\mu_1) = 8, \deg(\mu_1) = 7;$$

$$\mu_2 = 2 \times [0, 0], \dim(\mu_2) = 6, \deg(\mu_2) = 8.$$

## B Computer calculations

We will describe here the computer programs used in the calculations.<sup>8</sup>

1. We calculate the differential  $d : V \rightarrow S \otimes S$  (Eq. 2) using Gamma [16].
2. We use Macaulay2 [10] to calculate the Poincare (Hilbert) series  $P_n(\tau) = \sum_k \dim^n M_k \tau^k$  of  $R$ -module  ${}^n M = \sum_k H^{k,n}$ . Here  $R = \mathbb{C}[t^1, \dots, t^\alpha, \dots] = \sum_m \text{Sym}^m S$ . We calculate generators of this module and generators of free resolution

$$\cdots \rightarrow {}^n M_i \rightarrow \cdots \rightarrow {}^n M_1 \rightarrow {}^n M_0 \rightarrow {}^n M \rightarrow 0$$

where  ${}^n M_i = {}^n \mu_i \otimes R$ .

Input:

Coefficients  $\Gamma_{\alpha\beta}^m$  in the differential, the number of Greek indices ( $\dim S$ ), the number of Roman indices ( $\dim V$ ).

Output:

Poincare (Hilbert) series,  
number of generators of  ${}^n M$  and the number of them,  
number of generators of  ${}^n \mu_i$  having given degree.

3. Using LiE, we decompose  $\text{Sym}^m S \otimes \wedge^k V$  into irreducible representation of  $Aut$ . Applying principle of maximal propagation and resolving the ambiguities from the information about Poincare series we obtain decomposition of cohomology into irreducible representation for  $k \geq 20$ .
4. We make a conjecture of the decomposition of  $H^{k,n}$  into irreducible representation for arbitrary  $k$  using the information from the step 3. We prove that our conjecture gives the right Poincare series using Weyl dimension formula.<sup>9</sup>

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<sup>8</sup>The detailed codes are provided here: <http://lifshitz.ucdavis.edu/~rxu/code/cohom/>

<sup>9</sup>The Mathematica code for 10D case is provided here: <http://lifshitz.ucdavis.edu/~rxu/code/cohom/dim10dredux.nb>

5. We make a conjecture about cohomology generators using the information about their numbers and dimension from Macaulay2 [10] and the information from the steps 3 and 4. We prove that our formulas give cocycles using Gamma [16].
6. We use the formula Eq. 109 to get the decomposition of generators of free resolution into irreducible representation.

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# Homology of Lie algebra of supersymmetries and of super Poincaré Lie algebra\*

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**Abstract** We study the homology and cohomology groups of the super Lie algebra of supersymmetries and of the super Poincaré Lie algebra in various dimensions. We give complete answers for (non-extended) supersymmetry in all dimensions  $\leq 11$ . For dimensions  $D = 10, 11$  we describe also the cohomology of reduction of supersymmetry Lie algebra to lower dimensions. Our methods can be applied to extended supersymmetry Lie algebra.

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# 1 Introduction

In the present paper we will analyze homology and cohomology groups of the super Lie algebras of supersymmetries and of super Poincaré Lie algebras. We came to this problem from studying supersymmetric deformations of maximally supersymmetric gauge theories [15]; however, this problem arises also in different situations, in particular, in supergravity [2]. In low dimensions it was studied in [5]. The cohomology of supersymmetry Lie algebras appeared in the analysis of supersymmetric invariants in [4] (it was denoted there by the symbol  $H_t^{p,q}$ ). Some of the cohomology groups calculated below appear also in pure spinor formalism of ten-dimensional supersymmetric gauge theory in [3]. The present paper does not contain physical applications of our results, however, it is clear that these results should be useful in physics. In particular, combining them with considerations of [4] one can analyze the structure of supersymmetric invariants in situations that were not considered in [4]. Using the results of present paper in combination with ideas of [15] one can construct supersymmetric deformations of ten-dimensional super Yang-Mills theory reduced to dimension  $d \leq 10$ .

Some of the results of the present paper were derived by more elementary methods in our previous paper [16].

Let us recall the definition of Lie algebra cohomology (see [9] for more detail). We start with a super Lie algebra  $\mathcal{G}$  with generators  $e_A$  and structure constants  $f_{AB}^K$ . We introduce ghost variables  $C^A$  with parity opposite to the parity of generators  $e_A$  and consider the algebra  $E$  of polynomial functions of these variables. (In a more invariant way we can say that  $E$  consists of polynomial functions on linear superspace  $\Pi\mathcal{G}$ ). The algebra  $E$  is graded by the degree of polynomial. We define a derivation  $d$  on  $E$  by the formula  $d = \frac{1}{2}f_{AB}^K C^A C^B \frac{\partial}{\partial C^K}$ .

This operator is a differential (i.e. it changes the parity and obeys  $d^2 = 0$ .) We define the cohomology of  $\mathcal{G}$  using this differential:

$$H^\bullet(\mathcal{G}) = \text{Ker}d / \text{Im}d.$$

The definition of homology of  $\mathcal{G}$  is dual to the definition of cohomology: instead of  $E$  we consider its dual space  $E^*$  that can be considered as the space of functions of dual ghost variables  $C_A$ ; the differential  $\partial$  on  $E^*$  is defined as an operator adjoint to  $d$ . The homology  $H_\bullet(\mathcal{G})$  is dual to the cohomology  $H^\bullet(\mathcal{G})$ . We will work with cohomology, but our results can be interpreted in the language of homology.

Notice that we can multiply cohomology classes, i.e.  $H^\bullet(\mathcal{G})$  is an algebra.

The group  $Aut(\mathcal{G})$  of automorphisms of  $\mathcal{G}$  acts on  $E$  and commutes with the differential, therefore it acts also on homology and cohomology. We will be interested in this action. In other words we calculate cohomology as a representation of this group or as a representation of its Lie algebra  $aut(\mathcal{G})$  (as an  $aut(\mathcal{G})$ -module). For every graded module  $E$  we can define its Euler characteristic  $\chi(E)$  as a virtual module  $\sum (-1)^k E_k$  (as an alternating sum of its graded components in the sense of K-theory). The Euler characteristic of a graded differential module coincides with the Euler characteristic of its homology. This allows us to calculate the Euler characteristic of Lie algebra cohomology as a virtual representation (virtual  $aut(\mathcal{G})$ -module).<sup>1</sup> If the cohomology does not vanish only in one degree the Euler characteristic gives a complete answer for cohomology.<sup>2</sup>

The super Lie algebra of supersymmetries has odd generators  $e_\alpha$  and even generators  $P_m$ ; the only non-trivial commutation relation is

$$[e_\alpha, e_\beta]_+ = \Gamma_{\alpha\beta}^m P_m.$$

The coefficients in this relation are expressed in terms of Dirac Gamma matrices (see e.g. [6] for mathematical introduction).<sup>3</sup> The space  $E$  used in the definition

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<sup>1</sup>Instead of virtual modules we can talk about virtual representations of  $Aut(\mathcal{G})$  (elements of the representation ring). If the group  $Aut(\mathcal{G})$  is compact the representation ring can be identified with the ring of characters.

<sup>2</sup> This remark simplifies some of our calculations of cohomology of dimensional reductions of supersymmetry Lie algebra in dimensions 10 and 11. The most essential application of the Euler characteristic in our paper appears in the calculation of cohomology of ten-dimensional reduction of eleven-dimensional supersymmetry Lie algebra.

<sup>3</sup>The number of even generators is equal to the dimension of vector representation  $V$  of

of cohomology (cochain complex) consists here of polynomial functions of even ghost variables  $t^\alpha$  and odd ghost variables  $c^m$ ; the differential has the form

$$d = \frac{1}{2} \Gamma_{\alpha\beta}^m t^\alpha t^\beta \frac{\partial}{\partial c^m}. \quad (2)$$

The space  $E$  is double-graded (one can consider the degree with respect to  $t^\alpha$  and the degree with respect to  $c^m$ ). In more invariant form we can say that <sup>4</sup>

$$E = \bigoplus \text{Sym}^m S \otimes \Lambda^n V$$

and Gamma-matrices specify an intertwiner  $V \rightarrow \text{Sym}^2 S$ . The differential  $d$  maps  $\text{Sym}^m S \otimes \Lambda^n V$  into  $\text{Sym}^{m+2} S \otimes \Lambda^{n-1} V$ . The above description can be applied to any dimension and to any signature of the metric used in the definition of orthogonal group, however, the choice of the representation  $S$  is dimension-dependent. <sup>5</sup> The group  $\text{SO}(n)$  can be considered as a (subgroup) of the group of automorphisms of the supersymmetry Lie algebra and therefore it acts on its cohomology. The action of  $\text{SO}(n)$  is two-valued, hence it would be more precise to talk about action of its two-sheeted covering  $\text{Spin}(n)$  or about action of its Lie algebra  $\mathfrak{so}(n)$ . We will work with complex representations and complex Lie algebras; this does not change the cohomology

We will consider also homology and cohomology of the reduced Lie algebra of supersymmetries (or more precisely the Lie algebra of supersymmetries in dimension  $n$  reduced to the dimension  $d$ ). This algebra has the same odd the orthogonal group (to the dimension of space-time). The number of odd generators is the dimension of such a representation  $S$  of the orthogonal group that there exists an intertwiner  $V \rightarrow \text{Sym}^2 S$ .

<sup>4</sup>We use the notation  $\text{Sym}^m$  for symmetric tensor power and the notation  $\Lambda^n$  for exterior power

<sup>5</sup>Recall that the orthogonal group  $\text{SO}(2n)$  has two irreducible two-valued complex representations called semi-spin representations (left spinors and right spinors), the orthogonal group  $\text{SO}(2n+1)$  has one irreducible two-valued complex spin representation. One says that a complex representation is spinorial if it can be represented as a sum of spin or semi-spin representations. A real representation is spinorial if it becomes spinorial after extension of scalars to  $\mathbb{C}$ . (We follow the terminology of [6].) The representation  $S$  is spinorial. (See Section 6 for more detail about the representation  $S$ .)



generators  $e_\alpha$  as the Lie algebra of supersymmetries in dimension  $n$ , but only  $d \leq \dim V$  even generators  $P_1, \dots, P_d$ ; the commutation relations are the same as in unreduced algebra. In this case the cohomology is a representation of  $\text{Spin}(d) \times \text{Spin}(n-d)$ .

The double grading on  $E$  induces a double grading on cohomology. However, instead of the degrees  $m$  and  $n$  it is more convenient to use the degrees  $k = m + 2n$  and  $n$  because the differential preserves  $k$  and therefore the problem of calculation of cohomology can be solved for every  $k$  separately. It is important to notice that the differential commutes with multiplication by a polynomial in  $t^\alpha$ , therefore the cohomology is a module over the polynomial ring  $\mathbb{C}[t^1, \dots, t^\alpha, \dots]$ . (Moreover, it is an algebra over this ring.) The cohomology is infinite-dimensional as a vector space, but it has a finite number of generators as a  $\mathbb{C}[t^1, \dots, t^\alpha, \dots]$ -module (this follows from the fact that the polynomial ring is noetherian). One of the problems we would like to solve is the description of these generators. If cohomology classes of cocycles  $z_1, \dots, z_N$  generate the cohomology then every cohomology class can be represented by a cocycle of the form  $p_1 z_1 + \dots + p_N z_N$  where  $p_1, \dots, p_N$  belong to  $\mathbb{C}[t^1, \dots, t^\alpha, \dots]$ .

Notice that the cohomology of the Lie algebra of supersymmetries can be interpreted as the homology of the Koszul complex <sup>6</sup> corresponding to a sequence of functions  $f^m(t) = \frac{1}{2} \Gamma_{\alpha\beta}^m t^\alpha t^\beta$ . This allows us to use software [10] to calculate the dimensions of cohomology groups. However, we are interested in a more complicated problem- in the description of the decomposition of cohomology groups into a direct sum of irreducible representations of the group of automorphisms  $Aut$  or its Lie algebra  $aut$ . For small dimensions we use [8] for such calculations.

The paper is organized as follows. We start with the description of cohomology of the Lie algebra of supersymmetries in dimension 10 (Sec.2) and in dimension 11 (Sec.3). In the next sections we describe cohomology of dimensional reductions of the ten-dimensional algebra of supersymmetries (Sec.4) and

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<sup>6</sup>To define the homology of the Koszul complex corresponding to a sequence of functions  $f^1(t), \dots, f^n(t)$  one considers the differential  $d = f^m(t) \frac{\partial}{\partial \xi^m}$  where  $\xi^1, \dots, \xi^n$  are odd variables.

of the eleven-dimensional supersymmetries (Sec.5). Section 6 contains the results about Lie algebras of supersymmetries in dimensions  $\leq 9$ . Section 7 is devoted to the explanation of methods we are using. Section 8 is devoted to cohomology of the super Poincaré Lie algebra. The paper contains two appendices that will be omitted in the printed version. In Appendix A we describe the decomposition of the free resolution into a direct sum of representations of the automorphism group. Appendix B gives more detail about our calculations.

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## 2 D=10

We will start with the ten-dimensional case; in this case the spinorial representation in the definition of the Lie algebra of supersymmetries should be considered as one of two irreducible 16-dimensional representations of  $\text{Spin}(10)$  (in Minkowski space the spinors are Majorana-Weyl spinors). The Lie algebra of automorphisms *aut* is  $\mathfrak{so}(10)$ .

We will describe representations of the Lie algebra  $\mathfrak{so}(10)$  in the cohomology of the Lie algebra of supersymmetries in ten dimensions. As usual the representations are labeled by coordinates of their highest weight (see e.g. [18] for details). The vector representation  $V$  has the highest weight  $[1, 0, 0, 0, 0]$ , the irreducible spinor representations have highest weights  $[0, 0, 0, 0, 1], [0, 0, 0, 1, 0]$ ; we assume that the highest weight of  $S$  is  $[0, 0, 0, 0, 1]$ . The description of the graded component of the cohomology group with gradings  $k = m + 2n$  and  $n$  is

given by the formulas for  $H^{k,n}$  (for  $n \geq 6$ ,  $H^{k,n}$  vanishes)

$$H^{k,0} = [0, 0, 0, 0, k] \quad (3)$$

$$H^{k,1} = [0, 0, 0, 1, k-3] \quad (4)$$

$$H^{k,2} = [0, 0, 1, 0, k-6] \quad (5)$$

$$H^{k,3} = [0, 1, 0, 0, k-8] \quad (6)$$

$$H^{k,4} = [1, 0, 0, 0, k-10] \quad (7)$$

$$H^{k,5} = [0, 0, 0, 0, k-12] \quad (8)$$

The only special case is when  $k = 4$ , there is one additional term, a scalar, for  $H^{4,1}$ .

$$H^{4,1} = [0, 0, 0, 0, 0] \oplus [0, 0, 0, 1, 1] \quad (9)$$

The  $\text{SO}(10)$ -invariant part is in  $H^{0,0}$ ,  $H^{12,5}$ , and  $H^{4,1}$ .

The dimensions of these cohomology groups are encoded in series  $P_n(\tau) = \sum_k \dim H^{k,n} \tau^k$  (Poincaré series) that can be calculated by means of [10]:

$$P_0(\tau) = \frac{\tau^3 + 5\tau^2 + 5\tau + 1}{(1-\tau)^{11}}, \quad (10)$$

$$P_1(\tau) = (16\tau^3 + 35\tau^4 - \tau^5 + 55\tau^6 - 165\tau^7 + 330\tau^8 - 462\tau^9 + 462\tau^{10} - 330\tau^{11} + 165\tau^{12} - 55\tau^{13} + 11\tau^{14} - \tau^{15}) / (1-\tau)^{11}, \quad (11)$$

$$P_2(\tau) = (120\tau^6 - 120\tau^7 + 330\tau^8 - 462\tau^9 + 462\tau^{10} - 330\tau^{11} + 165\tau^{12} - 55\tau^{13} + 11\tau^{14} - \tau^{15}) / (1-\tau)^{11}, \quad (12)$$

$$P_3(\tau) = \frac{45\tau^8 + 65\tau^9 + 11\tau^{10} - \tau^{11}}{(1-\tau)^{11}}, \quad (13)$$

$$P_4(\tau) = \frac{10\tau^{10} + 34\tau^{11} + 16\tau^{12}}{(1-\tau)^{11}}, \quad (14)$$

$$P_5(\tau) = \frac{\tau^{12} + 5\tau^{13} + 5\tau^{14} + \tau^{15}}{(1-\tau)^{11}} \quad (15)$$

The cohomology regarded as a  $\mathbb{C}[t^1, \dots, t^\alpha, \dots]$ -module is generated by the

scalar considered as an element of  $H^{0,0}$  and by

$$\begin{aligned} [t^\alpha c_m \Gamma_{\alpha\beta}^m] &\in H^{3,1}, \\ [t^\alpha t^\beta c_m c_n \Gamma_{\alpha\beta}^{mnklr}] &\in H^{6,2}, \\ [t^\alpha t^\beta c_m c_n c_k \Gamma_{\alpha\beta}^{mnklr}] &\in H^{8,3}, \\ [t^\alpha t^\beta c_m c_n c_k c_l \Gamma_{\alpha\beta}^{mnklr}] &\in H^{10,4}, \\ [t^\alpha t^\beta c_m c_n c_k c_l c_r \Gamma_{\alpha\beta}^{mnklr}] &\in H^{12,5}. \end{aligned}$$

Here  $[a]$  denotes the cohomological class of cocycle  $a$ .

The GAMMA package [17] was used to verify that the expressions above are cocycles .

### 3 D=11

Now we consider the eleven-dimensional case; in this case the spinorial representation in the definition of supersymmetry Lie algebra should be considered as one irreducible 32-dimensional spinor representation of Spin(11) (Dirac spinors). As usual we work with complex representations and complex Lie algebras.

We will describe representations of  $Aut(\mathcal{G}) = \mathfrak{so}(11)$  in the the cohomology of the Lie algebra of supersymmetries. As usual the representations are labeled by their highest weight. The vector representation  $V$  has the highest weight  $[1, 0, 0, 0, 0]$ , the irreducible spinor representations have highest weights  $[0, 0, 0, 0, 1]$ . The description of graded component of cohomology group with gradings  $k = m + 2n$  and  $n$  is given by the formulas for  $H^{k,n}$  (for  $n \geq 3$ ,  $H^{k,n}$  vanishes)

$$H^{k,0} = \bigoplus_{i=0}^{[k/2]} [0, i, 0, 0, k - 2i] \quad (16)$$

$$H^{k,1} = \bigoplus_{i=0}^{[(k-4)/2]} [1, i, 0, 0, k - 4 - 2i] \quad (17)$$

$$H^{k,2} = \bigoplus_{i=0}^{[(k-6)/2]} [0, i, 0, 0, k - 6 - 2i] \quad (18)$$

The SO(11)-invariant part is in  $H^{0,0}$  and  $H^{6,2}$ .

The dimensions of these cohomology groups are encoded in Poincaré series:

$$P_0(\tau) = A(\tau) \quad (19)$$

$$P_1(\tau) = \frac{\tau^4(11 + 67\tau + 142\tau^2 + 142\tau^3 + 67\tau^4 + 11\tau^5)}{(1 - \tau)^{23}}, \quad (20)$$

$$P_2(\tau) = A(\tau)\tau^6 \quad (21)$$

where

$$A(\tau) = \frac{1 + 9\tau + 34\tau^2 + 66\tau^3 + 66\tau^4 + 34\tau^5 + 9\tau^6 + \tau^7}{(1 - \tau)^{23}} \quad (22)$$

The cohomology regarded as  $\mathbb{C}[t^1, \dots, t^\alpha, \dots]$ -module is generated by the scalar considered as an element of  $H^{0,0}$  and

$$\begin{aligned} [t^\alpha t^\beta c_m \Gamma_{\alpha\beta}^{mn}] &\in H^{4,1}, \\ [t^\alpha t^\beta c_m c_n \Gamma_{\alpha\beta}^{mn}] &\in H^{6,2}, \end{aligned}$$

## 4 Dimensional reduction from $D = 10$

Let us consider dimensional reductions of the ten-dimensional Lie algebra of supersymmetries. The reduction of  $\mathfrak{su}\mathfrak{sh}_{10}$  to  $r$  dimensions has 16 odd generators (supersymmetries) and  $r$  even generators (here  $0 \leq r \leq 10$ ). The corresponding differential has the form (2) where  $\Gamma_{\alpha\beta}^m$  are ten-dimensional Dirac matrices, Greek indices take 16 values as in the unreduced case, but Roman indices take only  $d$  values. The differential commutes with the (two-valued) action of the group  $\mathrm{SO}(r) \times \mathrm{SO}(10-r)$ , therefore this group acts on cohomology. The cohomology can be regarded as a module over  $\mathbb{C}[t^1, \dots, t^{16}]$ . Again cohomology is double graded; we use notation  $H^{k,n}$  for the component having degree  $m = k - 2n$  with respect to  $t$  and the degree  $n$  with respect to  $c$ . The symbol  $P_n(\tau)$  stands for the generating function  $P_n(\tau) = \sum_k \dim H^{k,n} \tau^k$  (for Poincaré series). We calculate the cohomology as a representation of the Lie algebra  $\mathfrak{so}(r) \times \mathfrak{so}(10-r)$  and describe elements that generate it as a  $\mathbb{C}[t^1, \dots, t^{16}]$ -module. (We characterize the representation by writing Dynkin labels of the first factor, then Dynkin labels of second factor.)

- $r = 9$ ,

$$H^{k,0} = [0, 0, 0, k], k \neq 2 \quad (23)$$

$$H^{k,1} = [0, 0, 1, k - 4] \quad (24)$$

$$H^{k,2} = [0, 1, 0, k - 6] \quad (25)$$

$$H^{k,3} = [1, 0, 0, k - 8] \quad (26)$$

$$H^{k,4} = [0, 0, 0, k - 10] \quad (27)$$

when  $k = 2$ ,

$$H^{2,0} = [0, 0, 0, 0] \oplus [0, 0, 0, 2] \quad (28)$$

Groups  $H^{k,n}$  with  $n \geq 5$  vanish. The  $\text{SO}(9)$ -invariant part is in  $H^{0,0}$ ,  $H^{10,4}$ , and  $H^{2,0}$ .

Generators

$$[t^\alpha t^\beta c_m \Gamma_{\alpha\beta}^{mnkl}] \in H^{4,1},$$

$$[t^\alpha t^\beta c_m c_n \Gamma_{\alpha\beta}^{mnkl}] \in H^{6,2},$$

$$[t^\alpha t^\beta c_m c_n c_k \Gamma_{\alpha\beta}^{mnkl}] \in H^{8,3},$$

$$[t^\alpha t^\beta c_m c_n c_k c_l \Gamma_{\alpha\beta}^{mnkl}] \in H^{10,4}.$$

Poincaré series

$$P_0(\tau) = (\tau^{13} - 11\tau^{12} + 55\tau^{11} - 165\tau^{10} + 330\tau^9 - 462\tau^8 + 462\tau^7 - 330\tau^6 + 165\tau^5 - 55\tau^4 + 10\tau^3 - 6\tau^2 - 5\tau - 1)/(-1 + \tau)^{11} \quad (29)$$

$$P_1(\tau) = (84\tau^4 - 156\tau^5 + 330\tau^6 - 462\tau^7 + 462\tau^8 - 330\tau^9 + 165\tau^{10} - 55\tau^{11} + 11\tau^{12} - \tau^{13})/(1 - \tau)^{11}, \quad (30)$$

$$P_2(\tau) = \frac{36\tau^6 + 36\tau^7}{(1 - \tau)^{11}}, \quad (31)$$

$$P_3(\tau) = \frac{9\tau^8 + 29\tau^9 + 11\tau^{10} - \tau^{11}}{(1 - \tau)^{11}}, \quad (32)$$

$$P_4(\tau) = \frac{\tau^{10} + 5\tau^{11} + 5\tau^{12} + \tau^{13}}{(1 - \tau)^{11}} \quad (33)$$

- $r = 8, k > 0$ ,

$$H^{k,0} = \bigoplus_{i=1}^{k-1} [0, 0, k-i, i, k-2i] \bigoplus_{i=0}^{[k/2]} [0, 0, k-2i, 0, k] \bigoplus_{i=0}^{[k/2]} [0, 0, 0, k-2i, -k], \quad (34)$$

$$H^{k,1} = \bigoplus_{i=0}^{k-4} [0, 1, k-4-i, i, k-4-2i], \quad (35)$$

$$H^{k,2} = \bigoplus_{i=0}^{k-6} [1, 0, k-6-i, i, k-6-2i], \quad (36)$$

$$H^{k,3} = \bigoplus_{i=0}^{k-8} [0, 0, k-8-i, i, k-8-2i] \quad (37)$$

Groups  $H^{k,n}$  with  $n \geq 4$  vanish. The  $\text{SO}(8) \times \text{SO}(2)$ -invariant part is in  $H^{0,0}$ , and  $H^{8,3}$ .

Generators:

$$\begin{aligned} [t^\alpha t^\beta c_m \Gamma_{\alpha\beta}^{mnk}] &\in H^{4,1}, \\ [t^\alpha t^\beta c_m c_n \Gamma_{\alpha\beta}^{mnk}] &\in H^{6,2}, \\ [t^\alpha t^\beta c_m c_n c_k \Gamma_{\alpha\beta}^{mnk}] &\in H^{8,3}, \end{aligned}$$

Poincaré series

$$P_0(\tau) = \frac{2\tau^5 - 6\tau^4 + 5\tau^3 - 7\tau^2 - 5\tau - 1}{(-1 + \tau)^{11}}, \quad (38)$$

$$P_1(\tau) = \frac{28\tau^4 + 12\tau^5 - 6\tau^6 + 2\tau^7}{(1 - \tau)^{11}}, \quad (39)$$

$$P_2(\tau) = \frac{8\tau^6 + 24\tau^7 + 6\tau^8 - 2\tau^9}{(1 - \tau)^{11}}, \quad (40)$$

$$P_3(\tau) = \frac{\tau^8 + 5\tau^9 + 5\tau^{10} + \tau^{11}}{(1 - \tau)^{11}} \quad (41)$$

- $r = 7$ ,

$$H^{k,0} = \bigoplus_{i=1}^{[k/2]} [0, i, k-2i, k-2i] \bigoplus_{i=0}^{[k/2]} [0, 0, k-2i, k] \quad (42)$$

$$H^{k,1} = \bigoplus_{i=0}^{[(k-4)/2]} [1, i, k-4-2i, k-4-2i] \quad (43)$$

$$H^{k,2} = \bigoplus_{i=0}^{[(k-6)/2]} [0, i, k-6-2i, k-6-2i] \quad (44)$$

Groups  $H^{k,n}$  with  $n \geq 3$  vanish. The  $\text{SO}(7) \times \text{SO}(3)$ -invariant part is in  $H^{0,0}$ , and  $H^{6,2}$ .

Generators:

$$\begin{aligned} [t^\alpha t^\beta c_m \Gamma_{\alpha\beta}^{mn}] &\in H^{4,1}, \\ [t^\alpha t^\beta c_m c_n \Gamma_{\alpha\beta}^{mn}] &\in H^{6,2}, \end{aligned}$$

Poincaré series

$$P_0(\tau) = \frac{5\tau^5 - 7\tau^4 + 8\tau^2 + 5\tau + 1}{(-1 + \tau)^{11}}, \quad (45)$$

$$P_1(\tau) = \frac{7\tau^4 + 19\tau^5 + \tau^6 - 3\tau^7}{(1 - \tau)^{11}}, \quad (46)$$

$$P_2(\tau) = \frac{\tau^6 + 5\tau^7 + 5\tau^8 + \tau^9}{(1 - \tau)^{11}} \quad (47)$$

- $r = 6$ ,

$$\begin{aligned} H^{k,0} &= \bigoplus_{i=0}^{[k/2]} \bigoplus_{j=0}^{k-2i} [j, i, k-j-2i, j, k-j-2i] \\ &\quad \bigoplus_{i=1}^{k-1} \bigoplus_{j=\max\{0, 2i-k\}}^{i-1} [j, 0, k-2i+j, i, k-i] \end{aligned} \quad (48)$$

$$H^{k,1} = \bigoplus_{i=0}^{[(k-4)/2]} \bigoplus_{j=0}^{k-4-2i} [j, i, k-4-j-2i, j, k-4-j-2i] \quad (49)$$

Groups  $H^{k,n}$  with  $n \geq 2$  vanish. The  $\text{SO}(6) \times \text{SO}(4)$ -invariant part is in  $H^{0,0}$ , and  $H^{4,1}$ .

Generators:

$$[t^\alpha t^\beta c_m \Gamma_{\alpha\beta}^m] \in H^{4,1} \quad (50)$$

Poincaré series

$$P_0(\tau) = \frac{4\tau^5 + 4\tau^4 - 5\tau^3 - 9\tau^2 - 5\tau - 1}{(-1 + \tau)^{11}}, \quad (51)$$

$$P_1(\tau) = \frac{\tau^4 + 5\tau^5 + 5\tau^6 + \tau^7}{(1 - \tau)^{11}} \quad (52)$$

- $r = 5$ ,

$$H^{k,0} = \bigoplus_{i=1}^{[k/2]} \bigoplus_{j=0}^{i-1} [j, k-2i, i, k-2i] \bigoplus_{i=0}^{[k/2]} \bigoplus_{j=0}^{[(k-2i)/2]} [i, k-2i-2j, i, k-2i-2j] \quad (53)$$



Groups  $H^{k,n}$  with  $n \geq 1$  vanish. The  $\text{SO}(5) \times \text{SO}(5)$ -invariant part lies in  $H^{k,0}$  where  $k$  is even.

Poincaré series

$$P_0(\tau) = \frac{(1+\tau)^5}{(1-\tau)^{11}} \quad (54)$$

- $r = 4$ ,

$$H^{k,0} = \bigoplus_{i=0}^{[k/2]} \bigoplus_{j=0}^{k-2i} (i+1) \times [j, k-2i-j, j, i, k-2i-j] \quad (55)$$

where the coefficient  $(i+1)$  is the multiplicity. Groups  $H^{k,n}$  with  $n \geq 1$  vanish. The  $\text{SO}(4) \times \text{SO}(6)$ -invariant part is in  $H^{0,0}$ .

Poincaré series

$$P_0(\tau) = \frac{(1+\tau)^4}{(1-\tau)^{12}} \quad (56)$$

- $r = 3$ ,

$$H^{k,0} = \bigoplus_{i=0}^{[k/2]} \bigoplus_{j=0}^i [k-2i, j, i-j, k-2i] \quad (57)$$

Groups  $H^{k,n}$  with  $n \geq 1$  vanish. The  $\text{SO}(3) \times \text{SO}(7)$ -invariant part is in  $H^{0,0}$ .

Poincaré series

$$P_0(\tau) = \frac{(1+\tau)^3}{(1-\tau)^{13}} \quad (58)$$

- $r = 2$ ,

$$H^{k,0} = \bigoplus_{i=0}^{[k/2]} \bigoplus_{j=0}^{k-2i} [i, 0, k-2i-j, j, k-2i-2j] \quad (59)$$

Groups  $H^{k,n}$  with  $n \geq 1$  vanish. The  $\text{SO}(2) \times \text{SO}(8)$ -invariant part is in  $H^{0,0}$ .

Poincaré series

$$P_0(\tau) = \frac{(1+\tau)^2}{(1-\tau)^{14}} \quad (60)$$

- $r = 1$ ,

$$H^{k,0} = \bigoplus_{i=0}^{[k/2]} [i, 0, 0, k-2i] \quad (61)$$

Groups  $H^{k,n}$  with  $n \geq 1$  vanish. The  $\text{SO}(1) \times \text{SO}(9)$ -invariant part is in  $H^{0,0}$ .

Poincaré series

$$P_0(\tau) = \frac{1 + \tau}{(1 - \tau)^{15}} \quad (62)$$

For  $r \leq 5$ , the cohomology is generated by the scalar 1.

## 5 Dimensional reduction from $D = 11$

Let us consider dimensional reductions of the eleven-dimensional Lie algebra of supersymmetries. The reduction of  $\mathfrak{su}\mathfrak{sh}_{11}$  to  $r$  dimensions has 32 odd generators (supersymmetries) and  $r$  even generators (here  $0 \leq r \leq 11$ ). The corresponding differential has the form (2) where  $\Gamma_{\alpha\beta}^m$  are eleven-dimensional Dirac matrices, Greek indices take 32 values as in unreduced case, but Roman indices take only  $d$  values. The differential commutes with action of the group  $\mathrm{SO}(r) \times \mathrm{SO}(11-r)$ , therefore this group acts on cohomology. The cohomology can be regarded as a module over  $\mathbb{C}[t^1, \dots, t^{32}]$ . Again cohomology is double graded; we use notation  $H^{k,n}$  for the component having degree  $m = k - 2n$  with respect to  $t$  and the degree  $n$  with respect to  $c$ . The symbol  $P_n(\tau)$  stands for the generating function  $P_n(\tau) = \sum_k \dim H^{k,n} \tau^k$  (for Poincaré series). We calculate the cohomology as a representation of the Lie algebra  $\mathfrak{so}(r) \times \mathfrak{so}(11-r)$  and describe elements that generate it as a  $\mathbb{C}[t^1, \dots, t^{32}]$ -module.

Let us start with calculation of Euler characteristic  $\chi(H^k)$  of cohomology  $H^k = \sum_n H^{k,n}$ . By general theorems this is a virtual  $\mathfrak{so}(r) \times \mathfrak{so}(11-r)$ -module

$$\sum_n (-1)^n \mathrm{Sym}^{k-2n} S \otimes \Lambda^n V \quad (63)$$

where  $S$  and  $V$  are considered as  $\mathfrak{so}(r) \times \mathfrak{so}(11-r)$ -modules.

Cohomology for  $r = 10$  are given by the formula

$$\begin{aligned}
H^{k,0} = & \bigoplus_{i=0}^{[k/2]} \bigoplus_{j=0}^{k-2i} \bigoplus_{l=0, l \neq 2}^{[ \frac{k-2i-j}{2} ]} [0, i, 0, j, k-2i-j-2l] \\
& \bigoplus_{i=1}^{[k/2]} \bigoplus_{j=0}^{[ \frac{k-2i}{2} ]} \bigoplus_{l=0}^{k-2i-2j} [i, j, 0, l, k-2i-2j-l] \\
& \bigoplus_{i=0}^{[ \frac{k-4}{2} ]} \bigoplus_{j=0}^{k-4-2i} [0, i, 0, j, k-4-2i-j], \tag{64}
\end{aligned}$$

$$H^{k,1} = \bigoplus_{i=0}^{[ \frac{k-4}{2} ]} \bigoplus_{j=0}^i \bigoplus_{l=0}^{k-4-2i} [j, i-j, 0, l, k-4-2i-l] \tag{65}$$

$$\tag{66}$$

Groups  $H^{k,n}$  with  $n \geq 2$  vanish. The  $\text{SO}(10)$ -invariant part is in  $H^{0,0}$  and  $H^{4,1}$ .

Generators

$$[t^\alpha t^\beta c_m \Gamma_{\alpha\beta}^m] \in H^{4,1} \tag{67}$$

Poincaré series

$$P_0(\tau) = \frac{(1+\tau)^{10}}{(1-\tau)^{22}} + \tau^4 A(\tau), \tag{68}$$

$$P_1(\tau) = \tau^4 A(\tau) \tag{69}$$

where  $A(\tau)$  is the Poincaré series given by Eq. 22.

For  $r \leq 9$  the groups  $H^{k,n}$  with  $n \geq 1$  vanish hence the Euler characteristic (63) gives a complete description of cohomology.

Poincaré series

$$P_0(\tau) = \frac{(1+\tau)^r}{(1-\tau)^{32-r}} \tag{70}$$

To find the  $\mathfrak{so}(r) \times \mathfrak{so}(11-r)$ -invariant part of  $H^{k,0}$  it is sufficient to solve this problem for the Euler characteristic. The conjectural answers (obtained by means of computations for  $k < 19$ ) are listed below.

$r = 1,$

$$[0, 0, 0, 0, 0] \in H^{4k,0} \tag{71}$$

$r = 2,$

$$([k/2] + 1) \times [0, 0, 0, 0, 0] \in H^{2k,0} \tag{72}$$

$r = 3,$

$$(k+1) \times [0, 0, 0, 0, 0] \in H^{4k,0} \quad (73)$$

$r = 4,$

$$\frac{(k+1)(k+2)}{2} \times [0, 0, 0, 0, 0] \in H^{4k,0} \quad (74)$$

$r = 5,$

$$\frac{([k/2]+1)([k/2]+2)([k/2]+3)}{6} \times [0, 0, 0, 0, 0] \in H^{2k,0} \quad (75)$$

$r = 6,$

$$\frac{([k/2]+1)([k/2]+2)}{2} \times [0, 0, 0, 0, 0] \in H^{2k,0} \quad (76)$$

$r = 7,$

$$(k+1) \times [0, 0, 0, 0, 0] \in H^{2k,0} \quad (77)$$

$r = 8,$

$$(2k+1) \times [0, 0, 0, 0, 0] \in H^{2k,0} \quad (78)$$

$r = 9,$

$$2 \times [0, 0, 0, 0, 0] \in H^{4k,0}, \quad (79)$$

$$[0, 0, 0, 0, 0] \in H^{4k+2,0} \quad (80)$$

where  $i \times [a, b, c, d, e]$  denotes the representation  $[a, b, c, d, e]$  with multiplicity  $i$  and  $[a]$  stands for the integer part of  $a$ .

## 6 Other dimensions

In this section we consider in detail cohomology of the Lie algebra of supersymmetries in dimensions  $< 10$ . Let us begin with some general discussion of supersymmetries in various dimensions (see [6] and [14] for more detail).

We will work with complex Lie algebras. Let us start with the description of the symmetric intertwiners  $\Gamma : S^* \otimes S^* \rightarrow V$  used in the construction of the supersymmetry Lie algebra in various dimensions (notice that in the construction of differential we use dual intertwiners). Recall that in even dimensions we have two irreducible spinorial representations  $\mathfrak{s}_l$  and  $\mathfrak{s}_r$ , in odd dimensions we have one irreducible spinorial representation  $\mathfrak{s}$ .

- $\dim V = 8n$

In this case we have intertwiners  $\gamma_l : \mathfrak{s}_l \otimes \mathfrak{s}_r \rightarrow V$  and  $\gamma_r : \mathfrak{s}_r \otimes \mathfrak{s}_l \rightarrow V$ .

$$S \cong S^* = \mathfrak{s}_l + \mathfrak{s}_r, \quad \Gamma = \gamma_l + \gamma_r, \quad \dim S = 16^n.$$

Automorphism Lie algebra  $\text{aut} = \mathfrak{so}(8n) \times \mathfrak{so}(2)$ .

- $\dim V = 8n + 1$

In this case we have one symmetric intertwiner  $\gamma : \mathfrak{s} \otimes \mathfrak{s} \rightarrow V$ .

$$S = S^* = \mathfrak{s}, \quad \Gamma = \gamma, \quad \dim S = 16^n.$$

Automorphism Lie algebra  $\text{aut} = \mathfrak{so}(8n + 1)$ .

- $\dim V = 8n + 2$

In this case we have symmetric intertwiners  $\gamma_l : \mathfrak{s}_l \otimes \mathfrak{s}_l \rightarrow V$  and  $\gamma_r : \mathfrak{s}_r \otimes \mathfrak{s}_r \rightarrow V$ .

There are two possible choices of  $S$  :

$$S = \mathfrak{s}_r, S^* = \mathfrak{s}_l, \Gamma = \gamma_l; \quad S = \mathfrak{s}_l, S^* = \mathfrak{s}_r, \Gamma = \gamma_r, \dim S = 16^n.$$

Automorphism Lie algebra  $\text{aut} = \mathfrak{so}(8n + 2)$ .

- $\dim V = 8n + 3$

In this case we have one symmetric intertwiner  $\gamma : \mathfrak{s} \otimes \mathfrak{s} \rightarrow V$ .

$$S = S^* = \mathfrak{s}, \quad \Gamma = \gamma, \quad \dim S = 2 \times 16^n.$$

Automorphism Lie algebra  $\text{aut} = \mathfrak{so}(8n + 3)$ .

- $\dim V = 8n + 4$

In this case we have intertwiners  $\gamma_l : \mathfrak{s}_l \otimes \mathfrak{s}_r \rightarrow V$  and  $\gamma_r : \mathfrak{s}_r \otimes \mathfrak{s}_l \rightarrow V$ .

$$S \cong S^* = \mathfrak{s}_l + \mathfrak{s}_r, \quad \Gamma = \gamma_l + \gamma_r, \quad \dim S = 4 \times 16^n.$$

Automorphism Lie algebra  $\text{aut} = \mathfrak{so}(8n + 4) \times \mathfrak{so}(2)$ .

- $\dim V = 8n + 5$

The intertwiner  $\gamma : \mathfrak{s} \otimes \mathfrak{s} \rightarrow V$  is antisymmetric.

$$S \cong S^* = \mathfrak{s} \otimes W, \Gamma = \gamma \otimes \omega, \dim S = 8 \times 16^n.$$

Here and later  $W$  stand for two-dimensional linear space with a symplectic form  $\omega$ . Automorphism Lie algebra  $aut = \mathfrak{so}(8n + 5) \times \mathfrak{sl}(2)$ .

- $\dim V = 8n + 6$

In this case we have antisymmetric intertwiners  $\gamma_l : \mathfrak{s}_l \otimes \mathfrak{s}_l \rightarrow V$  and  $\gamma_r : \mathfrak{s}_r \otimes \mathfrak{s}_r \rightarrow V$ . There are two possible choices of  $S$  :

$$S^* = \mathfrak{s}_l \otimes W, \Gamma = \gamma_l \otimes \omega; \quad S^* = \mathfrak{s}_r \otimes W, \Gamma = \gamma_r \otimes \omega, \dim S = 8 \times 16^n.$$

Automorphism Lie algebra  $aut = \mathfrak{so}(8n + 6) \times \mathfrak{sl}(2)$ .

- $\dim V = 8n + 7$

The intertwiner  $\gamma : \mathfrak{s} \otimes \mathfrak{s} \rightarrow V$  is antisymmetric.

$$S = S^* = \mathfrak{s} \otimes W, \Gamma = \gamma \otimes \omega, \dim S = 16 \times 16^n.$$

Automorphism Lie algebra  $aut = \mathfrak{so}(8n + 7) \times \mathfrak{sl}(2)$ .

One can consider also  $N$ -extended supersymmetry Lie algebra. This means that we should start with a reducible spinorial representation  $S_N$  (direct sum of  $N$  copies of the spinorial representation  $S$ ). Taking  $N$  copies of the intertwiner  $V \rightarrow \text{Sym}^2 S$  we obtain an intertwiner  $V \rightarrow \text{Sym}^2 S_N$ . We define the  $N$ -extended supersymmetry Lie algebra by means of this intertwiner. The Lie algebra acting on its cohomology acquires an additional factor  $\mathfrak{gl}(N)$ .

Notice that in the cases when there are two different possible choices of  $S$  (denoted by  $S_1$  and  $S_2$ ) one can talk about  $(N_1, N_2)$ -extended supersymmetry taking as a starting point a direct sum of  $N_1$  copies of  $S_1$  and  $N_2$  copies of  $S_2$ .

The description of cohomology of supersymmetry Lie algebras in dimensions 9, 8, 7 follows immediately from the description of cohomology of ten-dimensional

supersymmetry Lie algebra reduced to these dimensions. (Notice  $S$  has dimension 16 in all of these cases.)

We will describe the cohomology of the Lie algebra of supersymmetries in the six-dimensional case as representations of the Lie algebra  $\mathfrak{so}(6) \times \mathfrak{sl}(2)$ . The vector representation  $V$  of  $\mathfrak{so}(6)$  has the highest weight  $[1, 0, 0]$ , the irreducible spinor representations have highest weights  $[0, 0, 1]$ ,  $[0, 1, 0]$ ; we consider for definiteness  $\mathfrak{sl}$  with highest weight  $[0, 0, 1]$ . As a representation of  $\mathfrak{so}(6) \times \mathfrak{sl}(2)$  the representation  $V$  has the weight  $[1, 0, 0, 0]$  and the representation  $S = \mathfrak{sl} \otimes W$  has the weight  $[0, 0, 1, 1]$ . The description of the graded component of the cohomology group with gradings  $k = m + 2n$  and  $n$  is given by the formulas

$$H^{k,0} = [0, 0, k, k] \quad (81)$$

$$H^{k,1} = [1, 0, k-3, k-2] \quad (82)$$

$$H^{k,2} = [0, 1, k-6, k-4] \quad (83)$$

$$H^{k,3} = [0, 0, k-8, k-6] \quad (84)$$

The only special case is when  $k = 4$ , there is one additional term, a scalar, for  $H^{4,1}$ .

$$H^{4,1} = [0, 0, 0, 0] \oplus [1, 0, 1, 2] \quad (85)$$

For  $n \geq 4$ ,  $H^{k,n}$  vanishes. The  $\mathfrak{so}(6) \times \mathfrak{sl}(2)$ -invariant part is in  $H^{0,0}$ , and  $H^{4,1}$ .

The dimensions of the cohomology groups are encoded in Poincaré series:

$$P_0(\tau) = \frac{1 + 3\tau}{(1 - \tau)^5}, \quad (86)$$

$$P_1(\tau) = \frac{8\tau^3 + 6\tau^4 - 6\tau^5 + 10\tau^6 - 10\tau^7 + 5\tau^8 - \tau^9}{(1 - \tau)^5}, \quad (87)$$

$$P_2(\tau) = \frac{18\tau^6 - 10\tau^7 + 5\tau^8 - \tau^9}{(1 - \tau)^5}, \quad (88)$$

$$P_3(\tau) = \frac{3\tau^8 + \tau^9}{(1 - \tau)^5} \quad (89)$$

The cohomology considered as a  $\mathbb{C}[t^1, \dots, t^\alpha, \dots]$ -module is generated by the

scalar and

$$\begin{aligned} [t^\alpha c_m \Gamma_{\alpha\beta}^m] &\in H^{3,1}, \\ [t^\alpha t^\beta c_m c_n \Gamma_{\alpha\beta}^{mnk}] &\in H^{6,2}, \\ [t^\alpha t^\beta c_m c_n c_k \Gamma_{\alpha\beta}^{mnk}] &\in H^{8,3} \end{aligned}$$

Now we will describe the cohomology of the Lie algebra of supersymmetries in the five-dimensional case as representations of the Lie algebra  $\mathfrak{so}(5) \times \mathfrak{sl}(2)$ . The vector representation  $V$  of  $\mathfrak{so}(5)$  has the highest weight  $[1, 0]$ , the irreducible spinorial representation has highest weight  $[0, 1]$ . As a representation of  $\mathfrak{so}(5) \times \mathfrak{sl}(2)$  the representation  $V$  has the weight  $[1, 0, 0]$  and the representation  $S = \mathfrak{s} \otimes W$  has the weight  $[0, 1, 1]$ . The description of the graded component of the cohomology group with gradings  $k = m + 2n$  and  $n$  is given by the following formulas (for  $n \geq 3$ ,  $H^{k,n}$  vanishes)

$$H^{k,0} = [0, k, k] \quad (90)$$

$$H^{k,1} = [1, k-4, k-2] \quad (91)$$

$$H^{k,2} = [0, k-6, k-4] \quad (92)$$

The only special case is when  $k = 2$ , there is one additional term, a scalar, for  $H^{2,1}$ .

$$H^{2,1} = [0, 0, 0] \oplus [0, 2, 2] \quad (93)$$

The  $\mathfrak{so}(5) \times \mathfrak{sl}(2)$ -invariant part is in  $H^{0,0}$ , and  $H^{2,1}$ .

The dimensions of the cohomology groups are encoded in Poincaré series:

$$P_0(\tau) = \frac{1 + 3\tau + \tau^2 - 5\tau^3 + 10\tau^4 - 10\tau^5 + 5\tau^6 - \tau^7}{(1 - \tau)^5}, \quad (94)$$

$$P_1(\tau) = \frac{15\tau^4 - 11\tau^5 + 5\tau^6 - \tau^7}{(1 - \tau)^5}, \quad (95)$$

$$P_2(\tau) = \frac{3\tau^6 + \tau^7}{(1 - \tau)^5} \quad (96)$$

The cohomology regarded as a  $\mathbb{C}[t^1, \dots, t^\alpha, \dots]$ -module is generated by the



scalar considered as an element of  $H^{0,0}$  and

$$\begin{aligned} [t^\alpha t^\beta c_m \Gamma_{\alpha\beta}^{mn}] &\in H^{4,1}, \\ [t^\alpha t^\beta c_m c_n \Gamma_{\alpha\beta}^{mn}] &\in H^{6,2} \end{aligned}$$

In the four-dimensional case the representation  $S$  should be considered as 4-dimensional Dirac spinor.

We describe the cohomology of the Lie algebra of supersymmetries in the four-dimensional case as representations of the Lie algebra  $\mathfrak{so}(4)$ . As usual the representations are labeled by their highest weight. The vector representation  $V$  has the highest weight  $[1, 1]$ , the irreducible spinor representations have highest weights  $\mathfrak{s}_l = [0, 1]$ ,  $\mathfrak{s}_r = [1, 0]$ ; we assume that  $S = \mathfrak{s}_l + \mathfrak{s}_r = [0, 1] \oplus [1, 0]$ . The description of the graded component of the cohomology group with gradings  $k = m + 2n$  and  $n$  is given by the following formulas (for  $n \geq 6$ ,  $H^{k,n}$  vanishes)

$$H^{k,0} = [0, k] \oplus [k, 0] \quad (97)$$

$$H^{k,1} = [1, k-3] \oplus [k-3, 1] \quad (98)$$

$$H^{k,2} = [0, k-6] \oplus [k-6, 0], \quad (99)$$

The only special case is when  $k = 4$ , there is one additional term, a scalar, for  $H^{4,1}$ .

$$H^{4,1} = [0, 0] \oplus 2 \times [1, 1] \quad (100)$$

The  $\mathfrak{so}(4)$ -invariant part is in  $H^{0,0}$ ,  $H^{6,2}$ , and  $H^{4,1}$ .

The dimensions of these cohomology groups are encoded in Poincaré series:

$$P_0(\tau) = \frac{1 + 2\tau - \tau^2}{(1 - \tau)^2}, \quad (101)$$

$$P_1(\tau) = \frac{4\tau^3 + \tau^4 - 2\tau^5 + \tau^6}{(1 - \tau)^2}, \quad (102)$$

$$P_2(\tau) = \frac{2\tau^6}{(1 - \tau)^2} \quad (103)$$

The cohomology can be regarded as a  $\mathbb{C}[t^\alpha, t^{\dot{\alpha}}]$ -module where  $\alpha = 1, 2$ ,  $\dot{\alpha} = \dot{1}, \dot{2}$ . (Here  $t^\alpha$  transforms according to the representation  $[1, 0]$  and  $t^{\dot{\alpha}}$  transforms

according to the representation  $[0, 1]$ .) This module is generated by the scalar considered as an element of  $H^{0,0}$  and

$$\begin{aligned} [t^\alpha c_m \Gamma_{\alpha\dot{\beta}}^m] &\in H^{3,1}, \\ [t^{\dot{\alpha}} c_m \Gamma_{\dot{\alpha}\beta}^m] &\in H^{3,1}, \\ [t^\alpha t^{\dot{\beta}} c_m c_n \Gamma_{\alpha\dot{\beta}}^{mn}] &\in H^{6,2}, \\ [t^{\dot{\alpha}} t^\beta c_m c_n \Gamma_{\dot{\alpha}\beta}^{mn}] &\in H^{6,2} \end{aligned}$$

The cohomology generators in  $D = 4$  and  $D = 5$  were found by F. Brandt [5].

## 7 Calculations

We do calculation of Poincaré series applying [10]. However, the straightforward calculation is pretty lengthy with the computers we are using. Therefore for  $D = 10$  and  $D = 11$  we consider dimensional reduction to dimension  $r$  and we are using induction with respect to  $r$ .

Recall that the differential of the  $D$ -dimensional theory reduced to dimension  $r$  has the form

$$d_r = \sum_{1 \leq m \leq r} A^m \frac{\partial}{\partial c^m}. \quad (104)$$

where  $A^m = \frac{1}{2} \Gamma_{\alpha\beta}^m t^\alpha t^\beta$  and acts in the space  $E_r$  of polynomial functions of even ghosts  $t^\alpha$  and  $r$  odd ghosts  $c^1, \dots, c^r$ . We denote corresponding cohomology by  $H_r$ . Both  $E_r$  and  $H_r$  are bigraded by the degree of even ghosts  $m$  and degree of odd ghosts  $n$ , but it is simpler to work with grading with respect to  $k = m + 2n$  and  $n$ . An element of  $E_r$  can be represented in the form  $x + yc^r$  where  $x, y \in E_{r-1}$ . Notice that

$$d_r(x + yc^r) = d_{r-1}x + A^r y + d_{r-1}yc^r.$$

The operator of multiplication by  $A_m$  commutes with the differential, hence it induces a homomorphism

$$\sigma : H_{r-1} \rightarrow H_{r-1}.$$

Sending  $x \in E_{r-1}$  into  $x + 0c^m \in E_r$  (embedding  $E_{r-1}$  into  $E_r$ ) we obtain a homomorphism  $H_{r-1} \rightarrow H_r$ . Sending  $x + yc^m$  into  $y$  we get a homomorphism  $H_r \rightarrow H_{r-1}$ . It is easy to see that combining these homomorphisms we obtain an exact sequence

$$H_{r-1} \rightarrow H_{r-1} \rightarrow H_r \rightarrow H_{r-1} \rightarrow H_{r-1}$$

or, taking into account the gradings,

$$\cdots \rightarrow H_{r-1}^{k,n} \rightarrow H_{r-1}^{k+2,n} \rightarrow H_r^{k+2,n} \rightarrow H_{r-1}^{k,n-1} \rightarrow H_{r-1}^{k+2,n-1} \rightarrow \cdots \quad (105)$$

(This is the exact sequence of a pair  $(E_r, E_{r-1})$ ; we use the fact that  $E_r^{k,n}/E_{r-1}^{k,n} \cong E_{r-1}^{k-2,n-1}$ .) It follows immediately from this exact sequence that an isomorphism  $H_{r-1}^{k,n} = 0$  for  $n > n_{r-1}$  implies  $H_r^{k,n} = 0$  for  $n > n_r + 1$ . (In other words if  $n_r$  is the maximal degree of cohomology in  $r$ -dimensional reduction then  $n_{r+1} \leq n_r + 1$ .) Applying the exact sequence (105) to the case  $n = n_r$  and assuming that  $n_{r-1} < n_r$  we obtain an isomorphism between  $H_r^{k+2,n_r}$  and a subgroup of  $H_{r-1}^{k,n_{r-1}}$  (this isomorphism can be considered as an isomorphism of  $\mathfrak{so}(r-1)$ -representations). For dimensional reductions of  $D = 10$  and  $D = 11$  algebras of supersymmetries the dimensions of  $H_r^{k+2,n_r}$  and  $H_{r-1}^{k,n_{r-1}}$  coincide because Poincaré series are related by the formula  $P_{n_r} = \tau^2 P_{n_{r-1}}$ . If the homomorphism  $\sigma$  is injective we obtain a short exact sequence

$$0 \rightarrow H_{r-1}^{k,n} \rightarrow H_{r-1}^{k+2,n} \rightarrow H_r^{k+2,n} \rightarrow 0.$$

Calculations with [10] show that  $n_1 = \dots = n_5 = 0$  for  $D = 10$  and  $n_1 = \dots = n_9 = 0$  for  $D = 11$ . (It is sufficient to check that in corresponding dimensions the homomorphism  $\sigma$  is injective.)

To analyze  $r$ -dimensional reduction for  $r > 5$ ,  $D = 10$  we notice that  $d_r$  can be considered as a sum of differentials  $d'$  and  $d''$  where

$$d' = \sum_{1 \leq m \leq 5} A^m \frac{\partial}{\partial c^m},$$

$$d'' = \sum_{5 < m \leq r} A^m \frac{\partial}{\partial c^m}.$$

(For  $r > 9, D = 11$  one should replace 5 by 9.) These differentials anticommute; this allows us to use the spectral sequence of a bicomplex to calculate the cohomology of  $d_r$ . The spectral sequence of a bicomplex starts with cohomology  $H(d'', H(d'))$ . Taking into account that the cohomology  $H(d') = H(d_5)$  is concentrated in degree 0 (as the cohomology  $H_5$ ) we obtain that the spectral sequence terminates. This means that one can calculate the Poincaré series of  $d_r$  as the Poincaré series of  $H(d'', H(d'))$  using [10]. Again applying [10] we can obtain the information about generators of cohomology; this information is sufficient to express the generators in terms of Gamma-matrices.

To calculate the cohomology as a representation of the group of automorphisms we decompose each graded component  $E^{k,n} = \text{Sym}^{k-2n} S \otimes \wedge^n V$  of  $E$  into a direct sum of irreducible representations.

For example, for  $D = 10$  spacetime, we have the cochain complex

$$\begin{aligned}
0 \xleftarrow{d_0} \text{Sym}^k S \xleftarrow{d_1} \text{Sym}^{k-2} S \otimes V \xleftarrow{d_2} \text{Sym}^{k-4} S \otimes \wedge^2 V \\
\xleftarrow{d_3} \text{Sym}^{k-6} S \otimes \wedge^3 V \xleftarrow{d_4} \text{Sym}^{k-8} S \otimes \wedge^4 V \xleftarrow{d_5} \text{Sym}^{k-10} S \otimes \wedge^5 V \\
\xleftarrow{d_6} \text{Sym}^{k-12} S \otimes \wedge^6 V \xleftarrow{d_7} \text{Sym}^{k-14} S \otimes \wedge^7 V \xleftarrow{d_8} \text{Sym}^{k-16} S \otimes \wedge^8 V \\
\xleftarrow{d_9} \text{Sym}^{k-18} S \otimes \wedge^9 V \xleftarrow{d_{10}} \text{Sym}^{k-20} S \otimes \wedge^{10} V \xleftarrow{d_{11}} 0
\end{aligned} \tag{106}$$

where for  $\text{Sym}^m S \otimes \wedge^n V$ , the grading index  $k = m + 2n$  is preserved by  $d$ . All components of this complex can be regarded as representations of  $\mathfrak{so}(10)$ . We have

$$\begin{aligned}
S &= [0, 0, 0, 0, 1] \text{ (choosen) or } [0, 0, 0, 1, 0], \quad V = [1, 0, 0, 0, 0] \\
\wedge^2 V &= [0, 1, 0, 0, 0], \quad \wedge^3 V = [0, 0, 1, 0, 0], \\
\wedge^4 V &= [0, 0, 0, 1, 1], \quad \wedge^5 V = [0, 0, 0, 0, 2] \oplus [0, 0, 0, 2, 0], \\
\wedge^6 V &= \wedge^4 V, \quad \wedge^7 V = \wedge^3 V, \quad \wedge^8 V = \wedge^2 V, \quad \wedge^9 V = V, \quad \wedge^{10} V = [0, 0, 0, 0, 0],
\end{aligned} \tag{107}$$

$$\text{Sym}^k S = \bigoplus_{i=0}^{[k/2]} [i, 0, 0, 0, k-2i] \tag{108}$$

(see [16] for the decomposition of  $\text{Sym}^m S \otimes \wedge^n V$  and for a complete description of the action of the differential on irreducible components for supersymmetry

Lie algebra in 10D and 6D.)

By the Schur's lemma an intertwiner between irreducible representations (a homomorphism of simple modules) is either zero or an isomorphism. This means that an intertwiner between non-equivalent irreducible representations always vanishes. This observation permits us to calculate the contribution of every irreducible representation to the cohomology separately.

Let us fix an irreducible representation  $A$  and the number  $k$ . We will denote by  $\nu_n$  (or by  $\nu_n^k$  if it is necessary to show the dependence of  $k$ ) the multiplicity of  $A$  in  $E^{k,n} = \text{Sym}^{k-2n} S \otimes \Lambda^n V$ . The multiplicity of  $A$  in the image of  $d : E^{k,n} \rightarrow E^{k,n-1}$  will be denoted by  $\kappa_n$ , then the multiplicity of  $A$  in the kernel of this map is equal to  $\nu_n - \kappa_n$  and the multiplicity of  $A$  in the cohomology  $H^{k,n}$  is equal to  $h_n = \nu_n - \kappa_n - \kappa_{n+1}$ . It follows immediately that the multiplicity of  $A$  in the virtual representation  $\sum_n (-1)^n H^{k,n}$  (in the Euler characteristic) is equal to  $\sum (-1)^n \nu_n$ . It does not depend on  $\kappa_n$ , however, to calculate the cohomology completely we should know  $\kappa_n$ .

Let us consider as an example  $A = [0, 0, 0, 0, 0]$ , the scalar representation, for dimension  $D = 10$  and arbitrary  $k$ . For all  $k \neq 4, 12$ , we have  $\nu_i = 0$ . (For small  $k$  this can be obtained by means of LiE program [8].) For  $k = 4$ , we have all  $\nu_i$  vanish except  $\nu_1 = 1$ , hence all  $\kappa_i$  vanish. The multiplicity of  $[0, 0, 0, 0, 0]$  in  $H^{4,1}$  is equal to 1, and other cohomology  $H^{4,i}$  do not contain the scalar representation. For  $k = 12$ , all  $\nu_i$  vanish except  $\nu_5 = 1$ , hence  $H^{12,5}$  contains  $[0, 0, 0, 0, 0]$  with multiplicity 1, and  $H^{12,i}$  does not contain  $[0, 0, 0, 0, 0]$  for  $i \neq 5$ . This agrees with Eq. 9 and Eq. 8, respectively.

In many cases a heuristic calculation of cohomology can be based on a principle that the kernel should be as small as possible; in other words, the image should be as large as possible (this is an analog of the general rule of the physics of elementary particles: Everything happens unless it is forbidden). In [7] this is called *the principle of maximal propagation*.<sup>7</sup> Let us illustrate this principle

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<sup>7</sup> Notice that the principle of maximal propagation should be applied to the decomposition of cohomology into irreducible representations of the full automorphism group. Otherwise we do not use all available information.

in the case when  $k = 9$  and  $A = [0, 1, 0, 0, 1]$  in  $10D$ . In this case  $\nu_4 = 1$ ,  $\nu_3 = 3$ ,  $\nu_2 = 1$ . If we believe in the maximal propagation, then  $\kappa_3 = 1$ ,  $\kappa_4 = 1$ , thus we have  $\nu_3 - \kappa_3 - \kappa_4 = 1$ , and  $[0, 1, 0, 0, 1]$  contributes only to  $H^{9,3}$ .

Notice, that the principle of maximal propagation sometimes does not give a definite answer. For example, this is true in the case when  $k = 8$  in the dimension reduced to 7 from  $10D$ . Considering only the multiplicities of  $A = [0, 0, 2, 2]$ , we have the sequence  $0 \rightarrow 0 \rightarrow 2 \rightarrow 5 \rightarrow 3 \rightarrow 1$ . This sequence offers two distinct possibilities even under the assumption of maximal propagation. We can assume that the kernels of the differentials  $2 \rightarrow 5$  and  $5 \rightarrow 3$  are minimal. In this case  $h_0 = 1$ . Or we can start with the assumption that the kernel of differential  $3 \rightarrow 1$  is minimal, then the kernel of  $5 \rightarrow 3$  has multiplicity at least 3 and assuming that this multiplicity is equal to 3 we see under the assumption of minimality of the kernel of  $2 \rightarrow 5$  that the only non-trivial cohomology is  $h_2 = 1$ . (We can prove that the second position is the correct choice.) We see that the result of the method of maximal propagation depends on what differential we choose to start with. More generally, we should order the differentials in the complex in some way and apply the principle of maximal propagation using this ordering.

In cases we are interested in one can prove that the principle of maximal propagation augmented with information about Poincaré series and generators is free of ambiguities. In other words, this information permits us to resolve ambiguities in the application of this principle. Sometimes it is useful to apply the remark that multiplying a coboundary by a polynomial we again obtain a coboundary.

The only exception is the case of ten-dimensional reduction of the eleven-dimensional supersymmetry Lie algebra. In this case we use the isomorphism between  $H_{10}^{k,1}$  and  $H_{11}^{k-2,2}$  that was derived from Eq. 105. This is an isomorphism of  $\mathfrak{so}(10)$ -representations; it allows us to find the decomposition of  $H_{10}^{k,1}$  from decomposition of  $H_{11}^{k-2,2}$  in irreducible representations of  $\mathfrak{so}(11)$ . From the other side we can find the virtual  $\mathfrak{so}(10)$ -character of  $H_{10}^{k,0} - H_{10}^{k,1}$  (Euler characteristic); this allows us to finish the calculation.

Let  $\cdots \rightarrow M_n \rightarrow \cdots \rightarrow M_0 \rightarrow M \rightarrow 0$  denote the minimal free  $\sum \text{Sym}^m S$ -resolution of the module  $M = \sum_k H^{k,n}$ . Then every free module  $M_i$  in this resolution is a representation of the group of automorphisms  $Aut$ ; it is a tensor product of a finite dimensional graded representation  $\mu_i$  (the generators) and a representation of  $Aut$  in the polynomial algebra  $\sum \text{Sym}^m S$ . It is easy to find the dimensions of the  $Aut$ -modules  $\mu_i$  (the number of generators of  $M_i$ ) using [10]. Dimensions of the graded components of  $\mu_i$  can be found routinely using [10].

The information about the free resolution can be used to find the structure of  $Aut$ -module on  $\mu_i$  and therefore on  $M$ . However, we went in opposite direction: we used the information about the structure of  $Aut$ -module on  $M$  to find the structure of  $Aut$ -module on  $\mu_i$  using the formula <sup>8</sup>

$$\sum_i (-1)^i \mu_i = \left( \sum_k H^{k,n} \tau^k \right) \otimes \left( \sum (-\tau)^j \Lambda^j S \right). \quad (109)$$

The analysis of the resolution of the cohomology module is relegated to the appendix.

## 8 Homology of super Poincaré Lie algebra

The super Poincaré Lie algebra can be defined as the super Lie algebra spanned by the supersymmetry Lie algebra and the Lie algebra  $aut$  of its group of automorphisms  $Aut$ . <sup>9</sup>

To calculate the homology and cohomology of the super Poincaré Lie algebra we will use the following statement proved by J. Koszul [13] and by Hochschild and Serre [12]. (It follows from Hochschild-Serre spectral sequence constructed

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<sup>8</sup>This formula follows from well known  $K$ -theory relation

$$\left( \sum \tau^m \text{Sym}^m S \right) \left( \sum (-\tau)^j \Lambda^j S \right) = 1.$$

Taking into account that parity reversal transforms symmetric power into exterior power we can understand this relation in the framework of super algebra.

<sup>9</sup>Instead of the Lie algebra of automorphisms one can take any subalgebra. For example, we can take as a subalgebra the orthogonal Lie algebra

in [12])

Let  $\mathcal{P}$  denote a Lie algebra represented as a vector space as a direct sum of two subspaces  $\mathcal{L}$  and  $\mathcal{G}$ . We assume that  $\mathcal{G}$  is an ideal in  $\mathcal{P}$  and that  $\mathcal{L}$  is semisimple. It follows from the assumption that  $\mathcal{G}$  is an ideal that  $\mathcal{L}$  acts on  $\mathcal{G}$  and therefore on cohomology of  $\mathcal{G}$ ; the  $\mathcal{L}$ -invariant part of cohomology  $H^\bullet(\mathcal{G})$  will be denoted by  $H^\bullet(\mathcal{G})^\mathcal{L}$ . One can prove that

$$H^n(\mathcal{P}) = \sum_{p+q=n} H^p(\mathcal{L}) \otimes H^q(\mathcal{G})^\mathcal{L}.$$

This statement remains correct if  $\mathcal{P}$  is a super Lie algebra. We will apply it to the case when  $\mathcal{P}$  is the super Poincaré Lie algebra,  $\mathcal{G}$  is the Lie algebra of supersymmetries and  $\mathcal{L}$  is the Lie algebra of automorphisms or its semisimple subalgebra. (We are working with complex Lie algebras, but we can work with their real forms. The results do not change.)

Notice that it is easy to calculate the cohomology of the semisimple Lie algebra  $\mathcal{L}$ ; they are described by antisymmetric tensors on  $\mathcal{L}$  that are invariant with respect to the adjoint representation. One can say also that they coincide with de Rham cohomology of the corresponding compact Lie group. The Lie algebra cohomology of  $L = \mathfrak{so}_{10}$  with trivial coefficients and as well as De Rham cohomology of the compact Lie group  $\mathrm{SO}(10, \mathbb{R})$  is a Grassmann algebra with generators of dimension 3, 7, 11, 13 and 9. In general the cohomology of the group  $\mathrm{SO}(2r, \mathbb{R})$  is a Grassmann algebra with generators  $e_i$  having dimension  $4i - 1$  for  $i < r$  and the dimension  $2r - 1$  for  $i = r$ . The cohomology of the group  $\mathrm{SO}(2r + 1, \mathbb{R})$  is a Grassmann algebra with generators  $e_i$  having dimension  $4i - 1$  for  $i \leq r$ . The cohomology of the Lie algebra  $\mathfrak{sl}(n)$  coincide with the cohomology of compact Lie group  $\mathrm{SU}(n)$ ; they form a Grassmann algebra with generators of dimension 3, 5, ...,  $2n - 1$ .

As we have seen only the  $\mathcal{L}$ -invariant part of the cohomology of the Lie algebra of supersymmetries contributes to the cohomology of super Poincaré algebra. For  $D = 10$  this means that the only contribution comes from subspaces  $\mathrm{Sym}^m S \otimes \Lambda^n V$  having the following indices  $(m, n) = (0, 0)$ ,  $(m, n) = (2, 1)$  and  $(m, n) = (2, 5)$ , for  $D = 11$  the only contribution comes from  $(m, n) = (0, 0)$



and  $(m, n) = (2, 2)$ , for  $D = 6$  the only contribution comes from  $(m, n) = (0, 0)$  and  $(m, n) = (2, 1)$ . (Here  $m = k - 2n$  denotes the grading with respect to even ghosts  $t^\alpha$  and  $n$  the grading with respect to odd ghosts  $c_m$ .)

Cocycles representing cohomology classes of the super Poincaré algebra can be written in the form  $\rho \otimes h$ , where  $\rho$  is an invariant antisymmetric tensor with respect to the adjoint representation of  $aut$  and  $h$  is 1 or

$$t^\alpha t^\beta c_m c_n \Gamma_{\alpha\beta}^{mn} \text{ for } D = 11 \quad (110)$$

$$t^\alpha t^\beta c_m \Gamma_{\alpha\beta}^m, \quad t^\alpha t^\beta c_m c_n c_k c_l c_r \Gamma_{\alpha\beta}^{mnklr} \text{ for } D = 10 \quad (111)$$

$$t^\alpha t^\alpha, \quad t^\alpha t^\beta c_m c_n c_k c_l \Gamma_{\alpha\beta}^{mnkl} \text{ for } D = 9 \quad (112)$$

$$t^\alpha t^\beta c_m c_n c_k \Gamma_{\alpha\beta}^{mnk} \text{ for } D = 8 \quad (113)$$

$$t^\alpha t^\beta c_m c_n \Gamma_{\alpha\beta}^{mn} \text{ for } D = 7 \quad (114)$$

$$t^\alpha t^\beta c_m \Gamma_{\alpha\beta}^m, \quad t^\alpha t^\beta c_m c_n c_k \Gamma_{\alpha\beta}^{mnk} \text{ for } D = 6 \quad (115)$$

$$t^\alpha t^\beta c_m c_n \Gamma_{\alpha\beta}^{mn} \text{ for } D = 5 \quad (116)$$

$$t^\alpha t^{\dot{\beta}} c_m \Gamma_{\alpha\dot{\beta}}^m, t^{\dot{\alpha}} t^\beta c_m \Gamma_{\dot{\alpha}\beta}^m, t^\alpha t^{\dot{\beta}} c_m c_n \Gamma_{\alpha\dot{\beta}}^{mn}, t^{\dot{\alpha}} t^\beta c_m c_n \Gamma_{\dot{\alpha}\beta}^{mn} \text{ for } D = 4. \quad (117)$$

Here Greek indices (i.e. spinor indices) take values  $1, 2, \dots, \dim S$  and Roman indices (i.e. vector indices) take values  $1, 2, \dots, D$ , and  $\dim S$  is defined in Section.6. The only exception is for  $D = 4$ , the Greek indices  $\alpha, \beta$  take values  $1, 2$ , and the dotted Greek indices  $\dot{\alpha}, \dot{\beta}$  take values  $\dot{1}, \dot{2}$ . Notice that in these formulas Gamma matrices and summation range depend on the choice of dimension.

The general definition of the super Poincaré algebra can be applied also to the reduced supersymmetry Lie algebra. For  $D = 10$  and  $D = 11$  the role of the super Poincaré Lie algebra is played by the semidirect product of reduced supersymmetry Lie algebra and  $\mathfrak{so}(r) \times \mathfrak{so}(D - r)$ . The information about invariant elements provided in Sections 4 and 5 permits us to describe cohomology of this generalization of super Poincaré algebra.

## A Resolution of the cohomology modules

One can find a minimal free resolution of the  $R$ -module  $\sum_k H^{k,n} = M$ . (Here  $R = \mathbb{C}[t^1, \dots, t^\alpha, \dots] = \sum_m \text{Sym}^m S$ .) The reader may wish to consult [1] on this subject. The free resolution has the form

$$\cdots \rightarrow M_i \rightarrow \cdots \rightarrow M_0 \rightarrow M \rightarrow 0$$

where  $M_i = \mu_i \otimes R$ , and

$\mu_0$  - generators of  $M$ ;

$\mu_1$  - relations between generators of  $M$ ;

$\mu_2$  - relations between relations ;

$\dots$

### A.1 Resolution of the cohomology modules of dimensional reduction of 10D Lie algebra of supersymmetries

We give the structure of  $\mu_i$  as *aut*-module and its grading in the case of  $r$ -dimensional reduction of the ten-dimensional Lie algebra of supersymmetries (in the case when  $D = r + (10 - r)$ ). Recall that *aut* denotes the Lie algebra of the group of automorphisms *Aut*.

- D=10+0, n=0

$$\mu_0 = [0, 0, 0, 0, 0], \dim(\mu_0) = 1, \deg(\mu_0) = 0;$$

$$\mu_1 = [1, 0, 0, 0, 0], \dim(\mu_1) = 10, \deg(\mu_1) = 2;$$

$$\mu_2 = [0, 0, 0, 1, 0], \dim(\mu_2) = 16, \deg(\mu_2) = 3;$$

$$\mu_3 = [0, 0, 0, 0, 1], \dim(\mu_3) = 16, \deg(\mu_3) = 5;$$

$$\mu_4 = [1, 0, 0, 0, 0], \dim(\mu_4) = 10, \deg(\mu_4) = 6;$$

$$\mu_5 = [0, 0, 0, 0, 0], \dim(\mu_5) = 1, \deg(\mu_5) = 8.$$

- D=10+0, n=1

$$\mu_0 = [0, 0, 0, 1, 0], \dim(\mu_0) = 16, \deg(\mu_0) = 3;$$

$$\mu_1 = \mu_1' + \mu_1'',$$

$$\mu_1' = [0, 1, 0, 0, 0], \dim(\mu_1') = 45, \deg(\mu_1') = 4;$$

$$\mu_1'' = [0, 0, 0, 0, 1], \dim(\mu_1'') = 16, \deg(\mu_1'') = 5;$$

$$\mu_2 = 2 \times [0, 0, 1, 0, 0] + [1, 0, 0, 0, 0], \dim(\mu_2) = 250, \deg(\mu_2) = 6;$$

$$\mu_3 = [0, 0, 0, 1, 0] + [0, 1, 0, 1, 0] + [1, 0, 0, 0, 1], \dim(\mu_3) = 720, \deg(\mu_3) = 7;$$

$$\mu_4 = \mu_4' + \mu_4'',$$

$$\mu_4' = [0, 2, 0, 0, 0] + [1, 0, 0, 2, 0] + [2, 0, 0, 0, 0], \dim(\mu_4') = 1874, \deg(\mu_4') = 8;$$

$$\mu_4'' = [0, 0, 0, 0, 1], \dim(\mu_4'') = 16, \deg(\mu_4'') = 9;$$

$$\mu_5 = \mu_5' + \mu_5'',$$

$$\mu_5' = [0, 0, 0, 0, 1] + [0, 0, 0, 0, 1] + [0, 0, 0, 3, 0] + [1, 1, 0, 1, 0], \dim(\mu_5') = 4352, \deg(\mu_5') = 9;$$

$$\mu_5'' = [1, 0, 0, 0, 0], \dim(\mu_5'') = 9, \deg(\mu_5'') = 10;$$

$$\mu_6 = [0, 1, 0, 2, 0] + [2, 0, 1, 0, 0], \dim(\mu_6) = 8008, \deg(\mu_6) = 10;$$

$$\mu_7 = [1, 0, 1, 1, 0] + [3, 0, 0, 0, 1], \dim(\mu_7) = 11440, \deg(\mu_7) = 11;$$

$$\mu_8 = [0, 0, 2, 0, 0] + [2, 0, 0, 1, 1] + [4, 0, 0, 0, 0], \dim(\mu_8) = 12870, \deg(\mu_8) = 12;$$

$$\mu_9 = [1, 0, 1, 0, 1] + [3, 0, 0, 1, 0], \dim(\mu_9) = 11440, \deg(\mu_9) = 13;$$

$$\mu_{10} = [0, 1, 0, 0, 2] + [2, 0, 1, 0, 0], \dim(\mu_{10}) = 8008, \deg(\mu_{10}) = 14;$$

$$\mu_{11} = [0, 0, 0, 0, 3] + [1, 1, 0, 0, 1], \dim(\mu_{11}) = 4368, \deg(\mu_{11}) = 15;$$

$$\mu_{12} = [0, 2, 0, 0, 0] + [1, 0, 0, 0, 2], \dim(\mu_{12}) = 1820, \deg(\mu_{12}) = 16;$$

$$\mu_{13} = [0, 1, 0, 0, 1], \dim(\mu_{13}) = 560, \deg(\mu_{13}) = 17;$$

$$\mu_{14} = [0, 0, 1, 0, 0], \dim(\mu_{14}) = 120, \deg(\mu_{14}) = 18;$$

$$\mu_{15} = [0, 0, 0, 1, 0], \dim(\mu_{15}) = 16, \deg(\mu_{15}) = 19;$$

$$\mu_{16} = [0, 0, 0, 0, 0], \dim(\mu_{16}) = 1, \deg(\mu_{16}) = 20.$$

- D=10+0, n=2

$$\mu_0 = [0, 0, 1, 0, 0], \dim(\mu_0) = 120, \deg(\mu_0) = 6;$$

$$\mu_1 = [0, 0, 0, 1, 0] + [0, 1, 0, 1, 0] + [1, 0, 0, 0, 1], \dim(\mu_1) = 720, \deg(\mu_1) = 7;$$

$$\mu_2 = [0, 0, 0, 0, 0] + [0, 0, 0, 1, 1] + [0, 1, 0, 0, 0] + [0, 2, 0, 0, 0] + [1, 0, 0, 2, 0] + [2, 0, 0, 0, 0], \dim(\mu_2) = 2130, \deg(\mu_2) = 8;$$

$$\mu_3 = \mu_3' + \mu_3'',$$

$$\mu_3' = [0, 0, 0, 3, 0] + [1, 0, 0, 1, 0] + [1, 1, 0, 1, 0], \dim(\mu_3') = 4512, \deg(\mu_3') = 9;$$

$$\mu_3'' = [0, 0, 0, 0, 2] + [1, 0, 0, 0, 0], \dim(\mu_3'') = 136, \deg(\mu_3'') = 10;$$

$$\mu_4 = \mu_4' + \mu_4'',$$

$$\mu_4' = [0, 1, 0, 2, 0] + [2, 0, 1, 0, 0], \dim(\mu_4') = 8008, \deg(\mu_4') = 10;$$

$$\mu_4'' = [0, 0, 0, 1, 0] + [1, 0, 0, 0, 1], \dim(\mu_4'') = 160, \deg(\mu_4'') = 11;$$

$$\mu_5 = \mu_5' + \mu_5'',$$

$$\mu_5' = [1, 0, 1, 1, 0] + [3, 0, 0, 0, 1], \dim(\mu_5') = 11440, \deg(\mu_5') = 11;$$

$$\mu_5'' = [0, 1, 0, 0, 0], \dim(\mu_5'') = 45, \deg(\mu_5'') = 12;$$

$$\mu_6 = [0, 0, 2, 0, 0] + [2, 0, 0, 1, 1] + [4, 0, 0, 0, 0], \dim(\mu_6) = 12870, \deg(\mu_6) = 12;$$

$$\mu_7 = [1, 0, 1, 0, 1] + [3, 0, 0, 1, 0], \dim(\mu_7) = 11440, \deg(\mu_7) = 13;$$

$$\mu_8 = [0, 1, 0, 0, 2] + [2, 0, 1, 0, 0], \dim(\mu_8) = 8008, \deg(\mu_8) = 14;$$

$$\mu_9 = [0, 0, 0, 0, 3] + [1, 1, 0, 0, 1], \dim(\mu_9) = 4368, \deg(\mu_9) = 15;$$

$$\mu_{10} = [0, 2, 0, 0, 0] + [1, 0, 0, 0, 2], \dim(\mu_{10}) = 1820, \deg(\mu_{10}) = 16;$$

$$\mu_{11} = [0, 1, 0, 0, 1], \dim(\mu_{11}) = 560, \deg(\mu_{11}) = 17;$$

$$\mu_{12} = [0, 0, 1, 0, 0], \dim(\mu_{12}) = 120, \deg(\mu_{12}) = 18;$$

$$\mu_{13} = [0, 0, 0, 1, 0], \dim(\mu_{13}) = 16, \deg(\mu_{13}) = 19;$$

$$\mu_{14} = [0, 0, 0, 0, 0], \dim(\mu_{14}) = 1, \deg(\mu_{14}) = 20.$$

- D=10+0, n=3

$$\mu_0 = [0, 1, 0, 0, 0], \dim(\mu_0) = 45, \deg(\mu_0) = 8;$$

$$\mu_1 = [0, 0, 0, 0, 1] + [1, 0, 0, 1, 0], \dim(\mu_1) = 160, \deg(\mu_1) = 9;$$

$$\mu_2 = \mu_2' + \mu_2'',$$

$$\mu_2' = [0, 0, 0, 2, 0] + [1, 0, 0, 0, 0], \dim(\mu_2') = 136, \deg(\mu_2') = 10;$$

$$\mu_2'' = [1, 0, 0, 0, 1], \dim(\mu_2'') = 144, \deg(\mu_2'') = 11;$$

$$\mu_3 = [0, 0, 0, 0, 0] + [0, 0, 0, 1, 1] + [0, 1, 0, 0, 0] + [2, 0, 0, 0, 0], \dim(\mu_3) = 310, \deg(\mu_3) = 12;$$

$$\mu_4 = [0, 0, 0, 0, 1] + [1, 0, 0, 1, 0], \dim(\mu_4) = 160, \deg(\mu_4) = 13;$$

$$\mu_5 = [0, 0, 0, 1, 0], \dim(\mu_5) = 16, \deg(\mu_5) = 15;$$

$$\mu_6 = [0, 0, 0, 0, 0], \dim(\mu_6) = 1, \deg(\mu_6) = 16.$$

- D=10+0, n=4

$$\mu_0 = [1, 0, 0, 0, 0], \dim(\mu_0) = 10, \deg(\mu_0) = 10;$$

$$\mu_1 = \mu_1' + \mu_1'',$$

$$\mu_1' = [0, 0, 0, 1, 0], \dim(\mu_1') = 16, \deg(\mu_1') = 11;$$

$$\mu_1'' = [2, 0, 0, 0, 0], \dim(\mu_1'') = 54, \deg(\mu_1'') = 12;$$

$$\mu_2 = [0, 0, 0, 0, 1] + [1, 0, 0, 1, 0], \dim(\mu_2) = 160, \deg(\mu_2) = 13;$$

$$\mu_3 = [0, 0, 1, 0, 0] + [1, 0, 0, 0, 0], \dim(\mu_3) = 130, \deg(\mu_3) = 14;$$

$$\mu_4 = [0, 0, 0, 0, 0] + [0, 1, 0, 0, 0], \dim(\mu_4) = 46, \deg(\mu_4) = 16;$$

$$\mu_5 = [0, 0, 0, 0, 1], \dim(\mu_5) = 16, \deg(\mu_5) = 17.$$

- D=10+0, n=5

$$\mu_0 = [0, 0, 0, 0, 0], \dim(\mu_0) = 1, \deg(\mu_0) = 12;$$

$$\mu_1 = [1, 0, 0, 0, 0], \dim(\mu_1) = 10, \deg(\mu_1) = 14;$$

$$\mu_2 = [0, 0, 0, 1, 0], \dim(\mu_2) = 16, \deg(\mu_2) = 15;$$

$$\mu_3 = [0, 0, 0, 0, 1], \dim(\mu_3) = 16, \deg(\mu_3) = 17;$$

$$\mu_4 = [1, 0, 0, 0, 0], \dim(\mu_4) = 10, \deg(\mu_4) = 18;$$

$$\mu_5 = [0, 0, 0, 0, 0], \dim(\mu_5) = 1, \deg(\mu_5) = 20.$$

- D=9+1, n=0

$$\mu_0 = [0, 0, 0, 0], \dim(\mu_0) = 1, \deg(\mu_0) = 0;$$

$$\mu_1 = [1, 0, 0, 0], \dim(\mu_1) = 9, \deg(\mu_1) = 2;$$

$$\mu_2 = [0, 0, 1, 0] + [0, 1, 0, 0], \dim(\mu_2) = 120, \deg(\mu_2) = 4;$$

$$\mu_3 = [0, 0, 0, 1] + [0, 1, 0, 1] + [1, 0, 0, 1], \dim(\mu_3) = 576, \deg(\mu_3) = 5;$$

$$\mu_4 = [0, 0, 0, 0] + [0, 0, 0, 2] + [0, 2, 0, 0] + [1, 0, 0, 0] + [1, 0, 0, 2] + [1, 1, 0, 0] + \\ + [2, 0, 0, 0], \dim(\mu_4) = 1830, \deg(\mu_4) = 6;$$

$$\mu_5 = \mu_5' + \mu_5'',$$

$$\mu_5' = [0, 0, 0, 3] + [0, 1, 0, 1] + [1, 0, 0, 1] + [1, 1, 0, 1] + [2, 0, 0, 1], \dim(\mu_5') = 4368, \deg(\mu_5') = 7;$$

$$\mu_5'' = [0, 0, 0, 0], \dim(\mu_5'') = 1, \deg(\mu_5'') = 8;$$

$$\mu_6 = [0, 0, 1, 0] + [0, 1, 0, 0] + [0, 1, 0, 2] + [1, 0, 0, 2] + [1, 0, 1, 0] + [1, 1, 0, 0] + \\ + [2, 0, 1, 0] + [2, 1, 0, 0], \dim(\mu_6) = 8008, \deg(\mu_6) = 8;$$

$$\mu_7 = [0, 0, 0, 1] + [0, 0, 1, 1] + [0, 1, 0, 1] + [1, 0, 0, 1] + [1, 0, 1, 1] + [1, 1, 0, 1] + \\ + [2, 0, 0, 1] + [3, 0, 0, 1], \dim(\mu_7) = 11440, \deg(\mu_7) = 9;$$

$$\mu_8 = [0, 0, 0, 0] + [0, 0, 0, 2] + [0, 0, 1, 0] + [0, 0, 2, 0] + [0, 1, 1, 0] + [0, 2, 0, 0] + \\ + [1, 0, 0, 0] + [1, 0, 0, 2] + [1, 0, 1, 0] + [2, 0, 0, 0] + [2, 0, 0, 2] + [2, 0, 1, 0] + \\ + [3, 0, 0, 0] + [4, 0, 0, 0], \dim(\mu_8) = 12870, \deg(\mu_8) = 10;$$

$$\mu_9 = [0, 0, 0, 1] + [0, 0, 1, 1] + [0, 1, 0, 1] + [1, 0, 0, 1] + [1, 0, 1, 1] + [1, 1, 0, 1] + \\ + [2, 0, 0, 1] + [3, 0, 0, 1], \dim(\mu_9) = 11440, \deg(\mu_9) = 11;$$

$$\begin{aligned}
\mu_{10} &= [0, 0, 1, 0] + [0, 1, 0, 0] + [0, 1, 0, 2] + [1, 0, 0, 2] + [1, 0, 1, 0] + [1, 1, 0, 0] + \\
&\quad + [2, 0, 1, 0] + [2, 1, 0, 0], \dim(\mu_{10}) = 8008, \deg(\mu_{10}) = 12; \\
\mu_{11} &= [0, 0, 0, 3] + [0, 1, 0, 1] + [1, 0, 0, 1] + [1, 1, 0, 1] + [2, 0, 0, 1], \dim(\mu_{11}) = 4368, \deg(\mu_{11}) = 13; \\
\mu_{12} &= [0, 0, 0, 2] + [0, 2, 0, 0] + [1, 0, 0, 2] + [1, 1, 0, 0] + [2, 0, 0, 0], \dim(\mu_{12}) = 1820, \deg(\mu_{12}) = 14; \\
\mu_{13} &= [0, 1, 0, 1] + [1, 0, 0, 1], \dim(\mu_{13}) = 560, \deg(\mu_{13}) = 15; \\
\mu_{14} &= [0, 0, 1, 0] + [0, 1, 0, 0], \dim(\mu_{14}) = 120, \deg(\mu_{14}) = 16; \\
\mu_{15} &= [0, 0, 0, 1], \dim(\mu_{15}) = 16, \deg(\mu_{15}) = 17; \\
\mu_{16} &= [0, 0, 0, 0], \dim(\mu_{16}) = 1, \deg(\mu_{16}) = 18.
\end{aligned}$$

- D=9+1, n=1

$$\begin{aligned}
\mu_0 &= [0, 0, 1, 0], \dim(\mu_0) = 84, \deg(\mu_0) = 4; \\
\mu_1 &= [0, 0, 0, 1] + [0, 1, 0, 1] + [1, 0, 0, 1], \dim(\mu_1) = 576, \deg(\mu_1) = 5; \\
\mu_2 &= [0, 0, 0, 0] + [0, 0, 0, 2] + [0, 0, 1, 0] + [0, 1, 0, 0] + [0, 2, 0, 0] + [1, 0, 0, 0] + \\
&\quad + [1, 0, 0, 2] + [1, 1, 0, 0] + [2, 0, 0, 0], \dim(\mu_2) = 1950, \deg(\mu_2) = 6; \\
\mu_3 &= [0, 0, 0, 1] + [0, 0, 0, 3] + [0, 1, 0, 1] + 2 \times [1, 0, 0, 1] + [1, 1, 0, 1] + [2, 0, 0, 1], \\
&\quad \dim(\mu_3) = 4512, \deg(\mu_3) = 7; \\
\mu_4 &= \mu_4' + \mu_4'', \\
\mu_4' &= [0, 0, 1, 0] + [0, 1, 0, 0] + [0, 1, 0, 2] + [1, 0, 0, 2] + [1, 0, 1, 0] + [1, 1, 0, 0] + \\
&\quad + [2, 0, 0, 0] + [2, 0, 1, 0] + [2, 1, 0, 0], \dim(\mu_4') = 8052, \deg(\mu_4') = 8; \\
\mu_4'' &= [0, 0, 0, 1], \dim(\mu_4'') = 16, \deg(\mu_4'') = 9; \\
\mu_5 &= \mu_5' + \mu_5'', \\
\mu_5' &= [0, 0, 0, 1] + [0, 0, 1, 1] + [0, 1, 0, 1] + [1, 0, 0, 1] + [1, 0, 1, 1] + [1, 1, 0, 1] + \\
&\quad + [2, 0, 0, 1] + [3, 0, 0, 1], \dim(\mu_5') = 11440, \deg(\mu_5') = 9; \\
\mu_5'' &= [1, 0, 0, 0], \dim(\mu_5'') = 9, \deg(\mu_5'') = 10;
\end{aligned}$$

$$\begin{aligned}
\mu_6 &= [0, 0, 0, 0] + [0, 0, 0, 2] + [0, 0, 1, 0] + [0, 0, 2, 0] + [0, 1, 1, 0] + [0, 2, 0, 0] + \\
&\quad + [1, 0, 0, 0] + [1, 0, 0, 2] + [1, 0, 1, 0] + [2, 0, 0, 0] + [2, 0, 0, 2] + [2, 0, 1, 0] + \\
&\quad + [3, 0, 0, 0] + [4, 0, 0, 0], \dim(\mu_6) = 12870, \deg(\mu_6) = 10; \\
\mu_7 &= [0, 0, 0, 1] + [0, 0, 1, 1] + [0, 1, 0, 1] + [1, 0, 0, 1] + [1, 0, 1, 1] + [1, 1, 0, 1] + \\
&\quad + [2, 0, 0, 1] + [3, 0, 0, 1], \dim(\mu_7) = 11440, \deg(\mu_7) = 11; \\
\mu_8 &= [0, 0, 1, 0] + [0, 1, 0, 0] + [0, 1, 0, 2] + [1, 0, 0, 2] + [1, 0, 1, 0] + [1, 1, 0, 0] + \\
&\quad + [2, 0, 1, 0] + [2, 1, 0, 0], \dim(\mu_8) = 8008, \deg(\mu_8) = 12; \\
\mu_9 &= [0, 0, 0, 3] + [0, 1, 0, 1] + [1, 0, 0, 1] + [1, 1, 0, 1] + [2, 0, 0, 1], \dim(\mu_9) = 4368, \deg(\mu_9) = 13; \\
\mu_{10} &= [0, 0, 0, 2] + [0, 2, 0, 0] + [1, 0, 0, 2] + [1, 1, 0, 0] + [2, 0, 0, 0], \dim(\mu_{10}) = 1820, \deg(\mu_{10}) = 14; \\
\mu_{11} &= [0, 1, 0, 1] + [1, 0, 0, 1], \dim(\mu_{11}) = 560, \deg(\mu_{11}) = 15; \\
\mu_{12} &= [0, 0, 1, 0] + [0, 1, 0, 0], \dim(\mu_{12}) = 120, \deg(\mu_{12}) = 16; \\
\mu_{13} &= [0, 0, 0, 1], \dim(\mu_{13}) = 16, \deg(\mu_{13}) = 17; \\
\mu_{14} &= [0, 0, 0, 0], \dim(\mu_{14}) = 1, \deg(\mu_{14}) = 18.
\end{aligned}$$

- D=9+1, n=2

$$\begin{aligned}
\mu_0 &= [0, 1, 0, 0], \dim(\mu_0) = 36, \deg(\mu_0) = 6; \\
\mu_1 &= [0, 0, 0, 1] + [1, 0, 0, 1], \dim(\mu_1) = 144, \deg(\mu_1) = 7; \\
\mu_2 &= [0, 0, 0, 0] + [0, 0, 0, 2] + [1, 0, 0, 0] + [2, 0, 0, 0], \dim(\mu_2) = 180, \deg(\mu_2) = 8; \\
\mu_3 &= [0, 0, 0, 0] + [0, 0, 0, 2] + [1, 0, 0, 0] + [2, 0, 0, 0], \dim(\mu_3) = 180, \deg(\mu_3) = 10; \\
\mu_4 &= [0, 0, 0, 1] + [1, 0, 0, 1], \dim(\mu_4) = 144, \deg(\mu_4) = 11; \\
\mu_5 &= [0, 1, 0, 0], \dim(\mu_5) = 36, \deg(\mu_5) = 12.
\end{aligned}$$

- D=9+1, n=3

$$\mu_0 = [1, 0, 0, 0], \dim(\mu_0) = 9, \deg(\mu_0) = 8;$$

$$\mu_1 = \mu_1' + \mu_1'',$$



$$\mu_1' = [0, 0, 0, 1], \dim(\mu_1') = 16, \deg(\mu_1') = 9;$$

$$\mu_1'' = [2, 0, 0, 0], \dim(\mu_1'') = 44, \deg(\mu_1'') = 10;$$

$$\mu_2 = [0, 0, 0, 1] + [1, 0, 0, 1], \dim(\mu_2) = 144, \deg(\mu_2) = 11;$$

$$\mu_3 = [0, 0, 0, 0] + [0, 0, 1, 0] + [0, 1, 0, 0] + [1, 0, 0, 0], \dim(\mu_3) = 130, \deg(\mu_3) = 12;$$

$$\mu_4 = \mu_4' + \mu_4'',$$

$$\mu_4' = [0, 0, 0, 1], \dim(\mu_4') = 16, \deg(\mu_4') = 13;$$

$$\mu_4'' = [0, 1, 0, 0], \dim(\mu_4'') = 36, \deg(\mu_4'') = 14;$$

$$\mu_5 = [0, 0, 0, 1], \dim(\mu_5) = 16, \deg(\mu_5) = 15;$$

$$\mu_6 = [0, 0, 0, 0], \dim(\mu_6) = 1, \deg(\mu_6) = 16.$$

- D=9+1, n=4

$$\mu_0 = [0, 0, 0, 0], \dim(\mu_0) = 1, \deg(\mu_0) = 10;$$

$$\mu_1 = [0, 0, 0, 0] + [1, 0, 0, 0], \dim(\mu_1) = 10, \deg(\mu_1) = 12;$$

$$\mu_2 = [0, 0, 0, 1], \dim(\mu_2) = 16, \deg(\mu_2) = 13;$$

$$\mu_3 = [0, 0, 0, 1], \dim(\mu_3) = 16, \deg(\mu_3) = 15;$$

$$\mu_4 = [0, 0, 0, 0] + [1, 0, 0, 0], \dim(\mu_4) = 10, \deg(\mu_4) = 16;$$

$$\mu_5 = [0, 0, 0, 0], \dim(\mu_5) = 1, \deg(\mu_5) = 18.$$

- D=8+2, n=0

$$\mu_0 = [0, 0, 0, 0, 0], \dim(\mu_0) = 1, \deg(\mu_0) = 0;$$

$$\mu_1 = [1, 0, 0, 0, 0], \dim(\mu_1) = 8, \deg(\mu_1) = 2;$$

$$\mu_2 = [0, 1, 0, 0, 0], \dim(\mu_2) = 56, \deg(\mu_2) = 4;$$

$$\mu_3 = [0, 0, 0, 1, -1] + [0, 0, 1, 0, 1] + [1, 0, 0, 1, 1] + [1, 0, 1, 0, -1], \dim(\mu_3) = 128, \deg(\mu_3) = 5;$$

$$\begin{aligned} \mu_4 = & [0, 0, 0, 0, -2] + [0, 0, 0, 0, 2] + [0, 0, 0, 2, 2] + [0, 0, 2, 0, -2] + [1, 0, 0, 0, 0] + \\ & + [2, 0, 0, 0, -2] + [2, 0, 0, 0, 2], \dim(\mu_4) = 150, \deg(\mu_4) = 6; \end{aligned}$$

$$\mu_5 = \mu_5' + \mu_5'',$$

$$\mu_5' = [1, 0, 0, 1, 3] + [1, 0, 1, 0, -3], \dim(\mu_5') = 112, \deg(\mu_5') = 7;$$

$$\mu_5'' = [0, 0, 0, 0, 0], \dim(\mu_5'') = 1, \deg(\mu_5'') = 8;$$

$$\mu_6 = [0, 1, 0, 0, -4] + [0, 1, 0, 0, 4], \dim(\mu_6) = 56, \deg(\mu_6) = 8;$$

$$\mu_7 = [0, 0, 0, 1, -5] + [0, 0, 1, 0, 5], \dim(\mu_7) = 16, \deg(\mu_7) = 9;$$

$$\mu_8 = [0, 0, 0, 0, -6] + [0, 0, 0, 0, 6], \dim(\mu_8) = 2, \deg(\mu_8) = 10.$$

- D=8+2, n=1

$$\mu_0 = [0, 1, 0, 0, 0], \dim(\mu_0) = 28, \deg(\mu_0) = 4;$$

$$\mu_1 = [0, 0, 0, 1, -1] + [0, 0, 1, 0, 1] + [1, 0, 0, 1, 1] + [1, 0, 1, 0, -1], \dim(\mu_1) = 128, \deg(\mu_1) = 5;$$

$$\begin{aligned} \mu_2 = & [0, 0, 0, 0, -2] + [0, 0, 0, 0, 2] + [0, 0, 0, 2, 2] + [0, 0, 1, 1, 0] + [0, 0, 2, 0, -2] + \\ & + 2 \times [1, 0, 0, 0, 0] + [2, 0, 0, 0, -2] + [2, 0, 0, 0, 2], \dim(\mu_2) = 214, \deg(\mu_2) = 6; \end{aligned}$$

$$\mu_3 = \mu_3' + \mu_3'',$$

$$\mu_3' = [0, 0, 0, 1, 1] + [0, 0, 1, 0, -1] + [1, 0, 0, 1, 3] + [1, 0, 1, 0, -3], \dim(\mu_3') = 128, \deg(\mu_3') = 7;$$

$$\mu_3'' = [0, 0, 0, 0, 0] + [0, 0, 0, 2, 0] + [0, 0, 2, 0, 0] + [2, 0, 0, 0, 0], \dim(\mu_3'') = 106, \deg(\mu_3'') = 8;$$

$$\mu_4 = \mu_4' + \mu_4'',$$

$$\mu_4' = [0, 1, 0, 0, -4] + [0, 1, 0, 0, 4], \dim(\mu_4') = 56, \deg(\mu_4') = 8;$$

$$\mu_4'' = [0, 0, 0, 1, -1] + [0, 0, 1, 0, 1] + [1, 0, 0, 1, 1] + [1, 0, 1, 0, -1], \dim(\mu_4'') = 128, \deg(\mu_4'') = 9;$$

$$\mu_5 = \mu_5' + \mu_5'',$$

$$\mu_5' = [0, 0, 0, 1, -5] + [0, 0, 1, 0, 5], \dim(\mu_5') = 16, \deg(\mu_5') = 9;$$

$$\mu_5'' = [0, 1, 0, 0, -2] + [0, 1, 0, 0, 2] + [1, 0, 0, 0, 0], \dim(\mu_5'') = 64, \deg(\mu_5'') = 10;$$

$$\mu_6 = \mu_6' + \mu_6'',$$

$$\mu_6' = [0, 0, 0, 0, -6] + [0, 0, 0, 0, 6], \dim(\mu_6') = 2, \deg(\mu_6') = 10;$$

$$\mu_6'' = [0, 0, 0, 1, -3] + [0, 0, 1, 0, 3], \dim(\mu_6'') = 16, \deg(\mu_6'') = 11;$$

$$\mu_7 = [0, 0, 0, 0, -4] + [0, 0, 0, 0, 4], \dim(\mu_7) = 2, \deg(\mu_7) = 12.$$

- D=8+2, n=2

$$\mu_0 = [1, 0, 0, 0, 0], \dim(\mu_0) = 8, \deg(\mu_0) = 6;$$

$$\mu_1 = \mu_1' + \mu_1'',$$

$$\mu_1' = [0, 0, 0, 1, 1] + [0, 0, 1, 0, -1], \dim(\mu_1') = 16, \deg(\mu_1') = 7;$$

$$\mu_1'' = [2, 0, 0, 0, 0], \dim(\mu_1'') = 35, \deg(\mu_1'') = 8;$$

$$\mu_2 = \mu_2' + \mu_2'',$$

$$\mu_2' = [0, 0, 0, 0, 0], \dim(\mu_2') = 1, \deg(\mu_2') = 8;$$

$$\mu_2'' = [0, 0, 0, 1, -1] + [0, 0, 1, 0, 1] + [1, 0, 0, 1, 1] + [1, 0, 1, 0, -1], \dim(\mu_2'') = 128, \deg(\mu_2'') = 9;$$

$$\begin{aligned} \mu_3 = & [0, 0, 0, 0, -2] + [0, 0, 0, 0, 2] + [0, 0, 1, 1, 0] + [0, 1, 0, 0, -2] + [0, 1, 0, 0, 2] + \\ & + 2 \times [1, 0, 0, 0, 0], \dim(\mu_3) = 130, \deg(\mu_3) = 10; \end{aligned}$$

$$\mu_4 = \mu_4' + \mu_4'',$$

$$\mu_4' = [0, 0, 0, 1, -3] + [0, 0, 0, 1, 1] + [0, 0, 1, 0, -1] + [0, 0, 1, 0, 3], \dim(\mu_4') = 32, \deg(\mu_4') = 11;$$

$$\mu_4'' = [0, 1, 0, 0, 0], \dim(\mu_4'') = 28, \deg(\mu_4'') = 12;$$

$$\mu_5 = \mu_5' + \mu_5'',$$

$$\mu_5' = [0, 0, 0, 0, -4] + [0, 0, 0, 0, 4], \dim(\mu_5') = 2, \deg(\mu_5') = 12;$$

$$\mu_5'' = [0, 0, 0, 1, -1] + [0, 0, 1, 0, 1], \dim(\mu_5'') = 16, \deg(\mu_5'') = 13;$$

$$\mu_6 = [0, 0, 0, 0, -2] + [0, 0, 0, 0, 2], \dim(\mu_6) = 2, \deg(\mu_6) = 14.$$

- D=8+2, n=3

$$\mu_0 = [0, 0, 0, 0, 0], \dim(\mu_0) = 1, \deg(\mu_0) = 8;$$

$$\mu_1 = [0, 0, 0, 0, -2] + [0, 0, 0, 0, 2] + [1, 0, 0, 0, 0], \dim(\mu_1) = 10, \deg(\mu_1) = 10;$$

$$\mu_2 = [0, 0, 0, 1, 1] + [0, 0, 1, 0, -1], \dim(\mu_2) = 16, \deg(\mu_2) = 11;$$

$$\mu_3 = [0, 0, 0, 1, -1] + [0, 0, 1, 0, 1], \dim(\mu_3) = 16, \deg(\mu_3) = 13;$$

$$\mu_4 = [0, 0, 0, 0, -2] + [0, 0, 0, 0, 2] + [1, 0, 0, 0, 0], \dim(\mu_4) = 10, \deg(\mu_4) = 14;$$

$$\mu_5 = [0, 0, 0, 0, 0], \dim(\mu_5) = 1, \deg(\mu_5) = 16.$$

- D=7+3, n=0

$$\mu_0 = [0, 0, 0, 0], \dim(\mu_0) = 1, \deg(\mu_0) = 0;$$

$$\mu_1 = [1, 0, 0, 0], \dim(\mu_1) = 7, \deg(\mu_1) = 2;$$

$$\mu_2 = [0, 1, 0, 0] + [1, 0, 0, 0], \dim(\mu_2) = 28, \deg(\mu_2) = 4;$$

$$\mu_3 = \mu_3' + \mu_3'',$$

$$\mu_3' = [0, 0, 1, 1], \dim(\mu_3') = 16, \deg(\mu_3') = 5;$$

$$\mu_3'' = [0, 0, 0, 0] + [0, 0, 2, 0] + [2, 0, 0, 0], \dim(\mu_3'') = 63, \deg(\mu_3'') = 6;$$

$$\mu_4 = \mu_4' + \mu_4'',$$

$$\mu_4' = [0, 0, 0, 2], \dim(\mu_4') = 3, \deg(\mu_4') = 6;$$

$$\mu_4'' = [0, 0, 1, 1] + [1, 0, 1, 1], \dim(\mu_4'') = 112, \deg(\mu_4'') = 7;$$

$$\mu_5 = [0, 0, 0, 0] + [0, 1, 0, 2] + [1, 0, 0, 2], \dim(\mu_5) = 85, \deg(\mu_5) = 8;$$

$$\mu_6 = [0, 0, 1, 3], \dim(\mu_6) = 32, \deg(\mu_6) = 9;$$

$$\mu_7 = [0, 0, 0, 4], \dim(\mu_7) = 5, \deg(\mu_7) = 10.$$

- D=7+3, n=1

$$\mu_0 = [1, 0, 0, 0], \dim(\mu_0) = 7, \deg(\mu_0) = 4;$$

$$\mu_1 = \mu_1' + \mu_1'',$$

$$\mu_1' = [0, 0, 1, 1], \dim(\mu_1') = 16, \deg(\mu_1') = 5;$$

$$\mu_1'' = [2, 0, 0, 0], \dim(\mu_1'') = 27, \deg(\mu_1'') = 6;$$

$$\mu_2 = \mu_2' + \mu_2'',$$

$$\mu_2' = [0, 0, 0, 2], \dim(\mu_2') = 3, \deg(\mu_2') = 6;$$

$$\mu_2'' = [0, 0, 1, 1] + [1, 0, 1, 1], \dim(\mu_2'') = 112, \deg(\mu_2'') = 7;$$

$$\mu_3 = [0, 0, 0, 0] + [0, 0, 0, 2] + [0, 0, 2, 0] + [0, 1, 0, 2] + [1, 0, 0, 0] + [1, 0, 0, 2],$$

$$\dim(\mu_3) = 130, \deg(\mu_3) = 8;$$

$$\mu_4 = \mu_4' + \mu_4'',$$

$$\mu_4' = [0, 0, 1, 1] + [0, 0, 1, 3], \dim(\mu_4') = 48, \deg(\mu_4') = 9;$$

$$\mu_4'' = [0, 1, 0, 0], \dim(\mu_4'') = 21, \deg(\mu_4'') = 10;$$

$$\mu_5 = \mu_5' + \mu_5'',$$

$$\mu_5' = [0, 0, 0, 4], \dim(\mu_5') = 5, \deg(\mu_5') = 10;$$

$$\mu_5'' = [0, 0, 1, 1], \dim(\mu_5'') = 16, \deg(\mu_5'') = 11;$$

$$\mu_6 = [0, 0, 0, 2], \dim(\mu_6) = 3, \deg(\mu_6) = 12.$$

- D=7+3, n=2

$$\mu_0 = [0, 0, 0, 0], \dim(\mu_0) = 1, \deg(\mu_0) = 6;$$

$$\mu_1 = [0, 0, 0, 2] + [1, 0, 0, 0], \dim(\mu_1) = 10, \deg(\mu_1) = 8;$$

$$\mu_2 = [0, 0, 1, 1], \dim(\mu_2) = 16, \deg(\mu_2) = 9;$$

$$\mu_3 = [0, 0, 1, 1], \dim(\mu_3) = 16, \deg(\mu_3) = 11;$$

$$\mu_4 = [0, 0, 0, 2] + [1, 0, 0, 0], \dim(\mu_4) = 10, \deg(\mu_4) = 12;$$

$$\mu_5 = [0, 0, 0, 0], \dim(\mu_5) = 1, \deg(\mu_5) = 14.$$

- D=6+4, n=0

$$\mu_0 = [0, 0, 0, 0, 0], \dim(\mu_0) = 1, \deg(\mu_0) = 0;$$

$$\mu_1 = [0, 1, 0, 0, 0], \dim(\mu_1) = 6, \deg(\mu_1) = 2;$$

$$\mu_2 = [0, 0, 0, 0, 0] + [1, 0, 1, 0, 0], \dim(\mu_2) = 16, \deg(\mu_2) = 4;$$

$$\mu_3 = [0, 0, 0, 1, 1] + [0, 0, 2, 0, 0] + [0, 1, 0, 0, 0] + [2, 0, 0, 0, 0], \dim(\mu_3) = 30, \deg(\mu_3) = 6;$$

$$\mu_4 = \mu_4' + \mu_4'',$$

$$\mu_4' = [0, 0, 1, 1, 0] + [1, 0, 0, 0, 1], \dim(\mu_4') = 16, \deg(\mu_4') = 7;$$

$$\mu_4'' = [1, 0, 1, 0, 0], \dim(\mu_4'') = 15, \deg(\mu_4'') = 8;$$

$$\mu_5 = [0, 0, 1, 0, 1] + [1, 0, 0, 1, 0], \dim(\mu_5) = 16, \deg(\mu_5) = 9;$$

$$\mu_6 = [0, 0, 0, 1, 1], \dim(\mu_6) = 4, \deg(\mu_6) = 10.$$

- D=6+4, n=1

$$\begin{aligned}
\mu_0 &= [0, 0, 0, 0, 0], \dim(\mu_0) = 1, \deg(\mu_0) = 4; \\
\mu_1 &= [0, 0, 0, 1, 1] + [0, 1, 0, 0, 0], \dim(\mu_1) = 10, \deg(\mu_1) = 6; \\
\mu_2 &= [0, 0, 1, 1, 0] + [1, 0, 0, 0, 1], \dim(\mu_2) = 16, \deg(\mu_2) = 7; \\
\mu_3 &= [0, 0, 1, 0, 1] + [1, 0, 0, 1, 0], \dim(\mu_3) = 16, \deg(\mu_3) = 9; \\
\mu_4 &= [0, 0, 0, 1, 1] + [0, 1, 0, 0, 0], \dim(\mu_4) = 10, \deg(\mu_4) = 10; \\
\mu_5 &= [0, 0, 0, 0, 0], \dim(\mu_5) = 1, \deg(\mu_5) = 12.
\end{aligned}$$

- D=5+5, n=0

$$\begin{aligned}
\mu_0 &= [0, 0, 0, 0], \dim(\mu_0) = 1, \deg(\mu_0) = 0; \\
\mu_1 &= [1, 0, 0, 0], \dim(\mu_1) = 5, \deg(\mu_1) = 2; \\
\mu_2 &= [0, 2, 0, 0], \dim(\mu_2) = 10, \deg(\mu_2) = 4; \\
\mu_3 &= [0, 2, 0, 0], \dim(\mu_3) = 10, \deg(\mu_3) = 6; \\
\mu_4 &= [1, 0, 0, 0], \dim(\mu_4) = 5, \deg(\mu_4) = 8; \\
\mu_5 &= [0, 0, 0, 0], \dim(\mu_5) = 1, \deg(\mu_5) = 10.
\end{aligned}$$

- D=4+6, n=0

$$\begin{aligned}
\mu_0 &= [0, 0, 0, 0, 0], \dim(\mu_0) = 1, \deg(\mu_0) = 0; \\
\mu_1 &= [1, 1, 0, 0, 0], \dim(\mu_1) = 4, \deg(\mu_1) = 2; \\
\mu_2 &= [0, 2, 0, 0, 0] + [2, 0, 0, 0, 0], \dim(\mu_2) = 6, \deg(\mu_2) = 4; \\
\mu_3 &= [1, 1, 0, 0, 0], \dim(\mu_3) = 4, \deg(\mu_3) = 6; \\
\mu_4 &= [0, 0, 0, 0, 0], \dim(\mu_4) = 1, \deg(\mu_4) = 8.
\end{aligned}$$

- D=3+7, n=0

$$\mu_0 = [0, 0, 0, 0], \dim(\mu_0) = 1, \deg(\mu_0) = 0;$$

$$\mu_1 = [2, 0, 0, 0], \dim(\mu_1) = 3, \deg(\mu_1) = 2;$$

$$\mu_2 = [2, 0, 0, 0], \dim(\mu_2) = 3, \deg(\mu_2) = 4;$$

$$\mu_3 = [0, 0, 0, 0], \dim(\mu_3) = 1, \deg(\mu_3) = 6.$$

- D=2+8, n=0

$$\mu_0 = [0, 0, 0, 0, 0], \dim(\mu_0) = 1, \deg(\mu_0) = 0;$$

$$\mu_1 = [0, 0, 0, 0, -2] + [0, 0, 0, 0, 2], \dim(\mu_1) = 2, \deg(\mu_1) = 2;$$

$$\mu_2 = [0, 0, 0, 0, 0], \dim(\mu_2) = 1, \deg(\mu_2) = 4.$$

- D=1+9, n=0

$$\mu_0 = [0, 0, 0, 0], \dim(\mu_0) = 1, \deg(\mu_0) = 0;$$

$$\mu_1 = [0, 0, 0, 0], \dim(\mu_1) = 1, \deg(\mu_1) = 2.$$

## A.2 Resolution of the cohomology modules of 6D Lie algebra of supersymmetries

We now give the structure of  $\mu_i$  as an *aut*-module and its grading in the case of the six-dimensional Lie algebra of supersymmetries.

- D=6+0, n=0

$$\mu_0 = [0, 0, 0, 0], \dim(\mu_0) = 1, \deg(\mu_0) = 0;$$

$$\mu_1 = [0, 1, 0, 0], \dim(\mu_1) = 6, \deg(\mu_1) = 2;$$

$$\mu_2 = [1, 0, 0, 1], \dim(\mu_2) = 8, \deg(\mu_2) = 3;$$

$$\mu_3 = [0, 0, 0, 2], \dim(\mu_3) = 3, \deg(\mu_3) = 4.$$

- D=6+0, n=1

$$\mu_0 = [1, 0, 0, 1], \dim(\mu_0) = 8, \deg(\mu_0) = 3;$$

$$\mu_1 = [0, 0, 0, 2] + [1, 0, 1, 0], \dim(\mu_1) = 18, \deg(\mu_1) = 4;$$

$$\mu_2 = [0, 0, 2, 0] + [0, 1, 0, 2] + [2, 0, 0, 0], \dim(\mu_2) = 38, \deg(\mu_2) = 6;$$

$$\mu_3 = [0, 1, 1, 1] + [1, 0, 0, 1] + [1, 0, 0, 3], \dim(\mu_3) = 64, \deg(\mu_3) = 7;$$

$$\mu_4 = [0, 0, 0, 0] + [0, 0, 0, 4] + [0, 2, 0, 0] + [1, 0, 1, 2], \dim(\mu_4) = 71, \deg(\mu_4') = 8;$$

$$\mu_5 = [0, 0, 1, 3] + [1, 1, 0, 1], \dim(\mu_5) = 56, \deg(\mu_5) = 9;$$

$$\mu_6 = [0, 1, 0, 2] + [2, 0, 0, 0], \dim(\mu_6) = 28, \deg(\mu_6) = 10;$$

$$\mu_7 = [1, 0, 0, 1], \dim(\mu_7) = 8, \deg(\mu_7) = 11;$$

$$\mu_8 = [0, 0, 0, 0], \dim(\mu_8) = 1, \deg(\mu_8) = 12.$$

- D=6+0, n=2

$$\mu_0 = [0, 1, 0, 2], \dim(\mu_0) = 18, \deg(\mu_0) = 6;$$

$$\mu_1 = [0, 1, 1, 1] + [1, 0, 0, 1] + [1, 0, 0, 3], \dim(\mu_1) = 64, \deg(\mu_1) = 7;$$

$$\mu_2 = [0, 0, 0, 0] + [0, 0, 0, 2] + [0, 0, 0, 4] + [0, 2, 0, 0] + [1, 0, 1, 0] + [1, 0, 1, 2], \dim(\mu_2) = 89, \deg(\mu_2) = 8;$$

$$\mu_3 = [0, 0, 1, 1] + [0, 0, 1, 3] + [1, 1, 0, 1], \dim(\mu_3) = 64, \deg(\mu_3) = 9;$$

$$\mu_4 = [0, 1, 0, 2] + [2, 0, 0, 0], \dim(\mu_4) = 28, \deg(\mu_4') = 10;$$

$$\mu_5 = [1, 0, 0, 1], \dim(\mu_5) = 8, \deg(\mu_5) = 11;$$

$$\mu_6 = [0, 0, 0, 0], \dim(\mu_6) = 1, \deg(\mu_6) = 12.$$

- D=6+0, n=3

$$\mu_0 = [0, 0, 0, 2], \dim(\mu_0) = 3, \deg(\mu_0) = 8;$$

$$\mu_1 = [0, 0, 1, 1], \dim(\mu_1) = 8, \deg(\mu_1) = 9;$$

$$\mu_2 = [0, 1, 0, 0], \dim(\mu_2) = 6, \deg(\mu_2) = 10;$$

$$\mu_3 = [0, 0, 0, 0], \dim(\mu_3) = 1, \deg(\mu_3) = 12.$$



### A.3 Resolution of the cohomology modules of 5D Lie algebra of supersymmetries

We now give the structure of  $\mu_i$  as an *aut*-module and its grading in the case of the five-dimensional Lie algebra of supersymmetries.

- D=5+0, n=0

$$\mu_0 = [0, 0, 0], \dim(\mu_0) = 1, \deg(\mu_0) = 0;$$

$$\mu_1 = [1, 0, 0], \dim(\mu_1) = 5, \deg(\mu_1) = 2;$$

$$\mu_2 = [0, 2, 0] + [1, 0, 2], \dim(\mu_2) = 25, \deg(\mu_2) = 4;$$

$$\mu_3 = [0, 1, 1] + [0, 1, 3] + [1, 1, 1], \dim(\mu_3) = 56, \deg(\mu_3) = 5;$$

$$\mu_4 = [0, 0, 0] + [0, 0, 4] + [0, 2, 2] + [1, 0, 0] + [1, 0, 2] + [2, 0, 0], \dim(\mu_4) = 70, \deg(\mu_4') = 6;$$

$$\mu_5 = [0, 1, 1] + [0, 1, 3] + [1, 1, 1], \dim(\mu_5) = 56, \deg(\mu_5) = 7;$$

$$\mu_6 = [0, 0, 2] + [0, 2, 0] + [1, 0, 2], \dim(\mu_6) = 28, \deg(\mu_6) = 8;$$

$$\mu_7 = [0, 1, 1], \dim(\mu_7) = 8, \deg(\mu_7) = 9;$$

$$\mu_8 = [0, 0, 0], \dim(\mu_8) = 1, \deg(\mu_8) = 10.$$

- D=5+0, n=1

$$\mu_0 = [1, 0, 2], \dim(\mu_0) = 15, \deg(\mu_0) = 4;$$

$$\mu_1 = [0, 1, 1] + [0, 1, 3] + [1, 1, 1], \dim(\mu_1) = 56, \deg(\mu_1) = 5;$$

$$\mu_2 = [0, 0, 0] + [0, 0, 2] + [0, 0, 4] + [0, 2, 0] + [0, 2, 2] + [1, 0, 0] + [1, 0, 2] + [2, 0, 0], \dim(\mu_2) = 83, \deg(\mu_2) = 6;$$

$$\mu_3 = 2 \times [0, 1, 1] + [0, 1, 3] + [1, 1, 1], \dim(\mu_3) = 64, \deg(\mu_3) = 7;$$

$$\mu_4 = [0, 0, 0] + [0, 0, 2] + [0, 2, 0] + [1, 0, 2], \dim(\mu_4) = 29, \deg(\mu_4') = 8;$$

$$\mu_5 = [0, 1, 1], \dim(\mu_5) = 8, \deg(\mu_5) = 9;$$

$$\mu_6 = [0, 0, 0], \dim(\mu_6) = 1, \deg(\mu_6) = 10.$$

- D=5+0, n=2

$$\mu_0 = [0, 0, 2], \dim(\mu_0) = 3, \deg(\mu_0) = 6;$$

$$\mu_1 = [0, 1, 1], \dim(\mu_1) = 8, \deg(\mu_1) = 7;$$

$$\mu_2 = [0, 0, 0] + [1, 0, 0], \dim(\mu_2) = 6, \deg(\mu_2) = 8;$$

$$\mu_3 = [0, 0, 0], \dim(\mu_3) = 1, \deg(\mu_3) = 10.$$

#### A.4 Resolution of the cohomology modules of 4D Lie algebra of supersymmetries

We now give the structure of  $\mu_i$  as an *aut*-module and its grading in the case of the four-dimensional Lie algebra of supersymmetries.

- D=4+0, n=0

$$\mu_0 = [0, 0], \dim(\mu_0) = 1, \deg(\mu_0) = 0;$$

$$\mu_1 = [1, 1], \dim(\mu_1) = 4, \deg(\mu_1) = 2;$$

$$\mu_2 = [0, 1] + [1, 0], \dim(\mu_2) = 4, \deg(\mu_2) = 3;$$

$$\mu_3 = [0, 0], \dim(\mu_3) = 1, \deg(\mu_3) = 4.$$

- D=4+0, n=1

$$\mu_0 = [0, 1] + [1, 0], \dim(\mu_0) = 15, \deg(\mu_0) = 4;$$

$$\mu_1 = [0, 0] + [0, 2] + [2, 0], \dim(\mu_1) = 56, \deg(\mu_1) = 5;$$

$$\mu_2 = 2 \times [0, 0] + [1, 1], \dim(\mu_2) = 83, \deg(\mu_2) = 6;$$

$$\mu_3 = [0, 1] + [1, 0], \dim(\mu_3) = 64, \deg(\mu_3) = 7;$$

$$\mu_4 = [0, 0], \dim(\mu_4) = 29, \deg(\mu_4') = 8.$$

- D=4+0, n=2

$$\mu_0 = 2 \times [0, 0], \dim(\mu_0) = 3, \deg(\mu_0) = 6;$$

$$\mu_1 = [0, 1] + [1, 0], \dim(\mu_1) = 8, \deg(\mu_1) = 7;$$

$$\mu_2 = 2 \times [0, 0], \dim(\mu_2) = 6, \deg(\mu_2) = 8.$$

## B Computer calculations

We will describe here the computer programs used in the calculations.<sup>10</sup>

1. We calculate the differential  $d : V \rightarrow S \otimes S$  (Eq. 2) using Gamma [17].
2. We use Macaulay2 [10] to calculate the Poincaré (Hilbert) series  $P_n(\tau) = \sum_k \dim^n M_k \tau^k$  of  $R$ -module  ${}^n M = \sum_k H^{k,n}$ . Here  $R = \mathbb{C}[t^1, \dots, t^\alpha, \dots] = \sum_m \text{Sym}^m S$ . We calculate generators of this module and generators of free resolution

$$\cdots \rightarrow {}^n M_i \rightarrow \cdots \rightarrow {}^n M_1 \rightarrow {}^n M_0 \rightarrow {}^n M \rightarrow 0$$

where  ${}^n M_i = {}^n \mu_i \otimes R$ .

Input:

Coefficients  $\Gamma_{\alpha\beta}^m$  in the differential, the number of Greek indices ( $\dim S$ ), the number of Roman indices ( $\dim V$ ).

Output:

Poincaré (Hilbert) series,  
number of generators of  ${}^n M$  and the number of them,  
number of generators of  ${}^n \mu_i$  having given degree.

3. Using LiE, we decompose  $\text{Sym}^m S \otimes \wedge^k V$  into irreducible representation of  $Aut$ . Applying principle of maximal propagation and resolving the ambiguities from the information about Poincaré series we obtain decomposition of cohomology into irreducible representation for  $k \geq 20$ .
4. We make a conjecture of the decomposition of  $H^{k,n}$  into irreducible representation for arbitrary  $k$  using the information from the step 3. We prove that our conjecture gives the right Poincaré series using Weyl dimension formula.<sup>11</sup>

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<sup>10</sup>The detailed codes are provided here: <http://lifshitz.ucdavis.edu/~rxu/code/cohom/>

<sup>11</sup>The Mathematica code for 10D case is provided here: <http://lifshitz.ucdavis.edu/~rxu/code/cohom/dim10dredux.nb>

5. We make a conjecture about cohomology generators using the information about their numbers and dimension from Macaulay2 [10] and the information from the steps 3 and 4. We prove that our formulas give cocycles using Gamma [17].
6. We use the formula Eq. 109 to get the decomposition of generators of free resolution into irreducible representation.

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